A short post-contest overview of the problems

Below we present a sketch of solutions to the problems.

1 Springs *(S. Arthamonov)*

Consider five mass points on the plane located in four edges and the center of a square with side length L. Each mass point has mass m and is attached with springs to another mass points and four fixed points as shown on Fig. 1. All strings are non-stretched at the initial position. Each of the eight diagonal springs has stiffness k, whereas all four horizontal and vertical strings have stiffness $2k$.

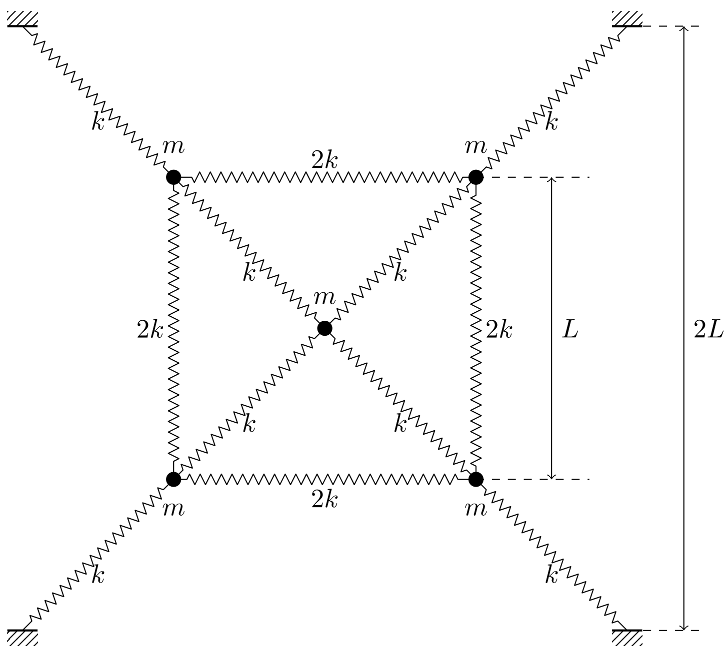

Figure 1: Springs arrangement

Considering the system as 2-dimensional:

1. Find all resonant frequencies for the classical system described above.
2. Consider quantum system with the same potential. Approximate five lowest energy levels for this system under assumption $mkL^4 \gg \hbar^2$.

Solution:

First, let us introduce the coordinate system with x-axis pointing to the upper right corner of Fig. 1 and y-axis pointing to the upper left corner. Let $\mathbf{r_1}, \ldots, \mathbf{r_4}$ be the vectors from the origin to the position of each of the four mass points in the corners of a square while $\mathbf{r_0}$ is the vector from the origin to the position of the mass point in the middle. Define x_i and y_i as follows

$$\mathbf{r_0} = (x_0, y_0)\,,$$
$$\mathbf{r_1} = \left(\frac{1}{\sqrt{2}} + x_1,\, y_1\right), \qquad \mathbf{r_2} = \left(x_2,\, \frac{1}{\sqrt{2}} + y_2\right),$$
$$\mathbf{r_3} = \left(-\frac{1}{\sqrt{2}} + x_3,\, y_3\right), \qquad \mathbf{r_4} = \left(x_4,\, -\frac{1}{\sqrt{2}} + y_4\right).$$

Potential energy is then given by

$$V = \sum_{i=1}^{4} \big(V_L(\mathbf{r_i}, \mathbf{r_{i+1}}) + V_S(\mathbf{r_i}, \mathbf{r_0}) + V_S(\mathbf{r_i}, \mathbf{R_i}) \big), \tag{1}$$

where $\mathbf{r_5} := \mathbf{r_1}$ and

$$V_L(\mathbf{a}, \mathbf{b}) = k\left(|\mathbf{a}-\mathbf{b}| - L\right)^2 \qquad V_S(\mathbf{a}, \mathbf{b}) = \frac{k}{2}\left(|\mathbf{a}-\mathbf{b}| - \frac{L}{\sqrt{2}}\right)^2.$$

Corresponding $\mathbf{R_i}$ are the fixed vectors

$$\mathbf{R_1} = \left(\sqrt{2}, 0\right), \qquad \mathbf{R_2} = \left(0, \sqrt{2}\right),$$
$$\mathbf{R_3} = \left(-\sqrt{2}, 0\right), \qquad \mathbf{R_4} = \left(0, -\sqrt{2}\right).$$

Decomposing potential (1) at $x_i = y_i = 0$ up to the bilinear terms we get

$$\begin{aligned} V_2 =& k(x_0^2 + x_2^2 + x_4^2 + y_0^2 + y_1^2 + y_3^2) + 2k(x_1^2 + x_3^2 + y_2^2 + y_4^2) + \\ &+ k(x_2y_1 - x_4y_1 + x_1y_2 - x_3y_2 - x_2y_3 + x_4y_3 - x_1y_4 + x_3y_4) \\ &- k\sum_{i<j}(x_ix_j + y_iy_j) \end{aligned}$$

One can present the quadratic terms in potential in terms of matrix $\mathbf{B}$ of bilinear form

$$V_2 = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_0 \end{pmatrix}^T \begin{pmatrix} k & -\frac{k}{2} & 0 & -\frac{k}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k}{2} & 2k & -\frac{k}{2} & 0 & -\frac{k}{2} & 0 & \frac{k}{2} & 0 & -\frac{k}{2} & 0 \\ 0 & -\frac{k}{2} & k & -\frac{k}{2} & 0 & \frac{k}{2} & 0 & -\frac{k}{2} & 0 & 0 \\ -\frac{k}{2} & 0 & -\frac{k}{2} & 2k & -\frac{k}{2} & 0 & -\frac{k}{2} & 0 & \frac{k}{2} & 0 \\ 0 & -\frac{k}{2} & 0 & -\frac{k}{2} & k & -\frac{k}{2} & 0 & \frac{k}{2} & 0 & 0 \\ 0 & 0 & \frac{k}{2} & 0 & -\frac{k}{2} & k & -\frac{k}{2} & 0 & -\frac{k}{2} & 0 \\ 0 & \frac{k}{2} & 0 & -\frac{k}{2} & 0 & -\frac{k}{2} & 2k & -\frac{k}{2} & 0 & -\frac{k}{2} \\ 0 & 0 & -\frac{k}{2} & 0 & \frac{k}{2} & 0 & -\frac{k}{2} & k & -\frac{k}{2} & 0 \\ 0 & -\frac{k}{2} & 0 & \frac{k}{2} & 0 & -\frac{k}{2} & 0 & -\frac{k}{2} & 2k & -\frac{k}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{k}{2} & 0 & -\frac{k}{2} & k \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_0 \end{pmatrix}$$

Matrix $\mathbf{B}$ has the following eigenvalues:

$$0, \quad k, \quad 2k, \quad 3k, \quad \frac{3+\sqrt{7}}{2}k, \quad \frac{3-\sqrt{7}}{2}k.$$

Since all point masses are equal, this corresponds to one zero mode and the following resonant frequencies:

$$\sqrt{\frac{2k}{m}}, \quad 2\sqrt{\frac{k}{m}}, \quad \sqrt{\frac{6k}{m}}, \quad \sqrt{\frac{(3+\sqrt{7})k}{m}}, \quad \sqrt{\frac{(3-\sqrt{7})k}{m}}.$$

Now, consider the quantum system with the same potential. Matrix $\mathbf{B}$ has a single eigenvector corresponding to the zero eigenvalue, namely

$$(0, 0, 1, 0, -1, -1, 0, 1, 0, 0)^T.$$

This corresponds to rotation of the square around the center. Under assumption $mkL^4 \gg \hbar^2$ the lowest energy levels would be the energy levels of the nonharmonic oscillator corresponding to this mode. To find the leading term in the effective potential of this oscillator lets find the curve in the coordinate space which corresponds to the slowest growth of the potential. From symmetry considerations one can conclude that it corresponds to simultaneous rotation of the four points in the corner of the square by the same angle θ and change their distance from the center to $R(\theta)$. Exact formula for the potential energy then reads

$$V_{rot}(R,\theta) = 4\frac{k}{2}\left(R - \frac{L}{\sqrt{2}}\right)^2 + 4\frac{2k}{k}\left(\sqrt{2}R - L\right)^2 + \\ + 4\frac{k}{2}\left(\sqrt{R^2\sin^2\theta + (\sqrt{2}L - R\cos\theta)^2} - \frac{1}{\sqrt{2}}L\right)^2.$$

Now, one should determine $R(\theta)$ by the condition that $V(R(\theta),\theta)$ is minimal for a fixed θ. Let's compute the leading fourth order in the potential. To do this, denote

$$R(\theta) = \frac{L}{\sqrt{2}} + c\,\theta^2 + \mathbf{O}(\theta^{\mathbf{3}}).$$

Then

$$V_{rot} = \left(12c^2k - 2\sqrt{2}c\,k\,L + k\,L^2\right)\theta^4 + \mathbf{O}(\theta^5).$$

This is minimized for $c = \frac{L}{6\sqrt{2}}$, and we have for effective potential

$$V_{eff} = \frac{5}{6}k\,L^2\theta^4.$$

So we have to estimate lowest five energy levels for the following Schrödinger equation

$$-\frac{\hbar^2}{4mL^2}\frac{\partial^2}{\partial\theta^2}\Psi(\theta) + \frac{5}{6}k\,L^2\theta^4\Psi(\theta) = E\Psi(\theta).$$

There are many nice approaches to the nonharmonic oscillators which can be found in the literature, below we present a very simple (and rough) solution. One can get an estimate of the energy levels using WKB approximation, indeed

$$\int_{-\theta_B(E_n)}^{\theta_B(E_n)} \sqrt{\frac{4mL^2E_n}{\hbar^2} - \frac{10mkL^4\theta^4}{3\hbar^2}}\,\mathrm{d}\theta = \pi + 2\pi n \tag{2}$$

where

$$\theta_B(E_n) = \sqrt[4]{\frac{6E_n}{5kL^2}}$$

The rough estimate for (2) would be

$$2\sqrt{\frac{4mL^2E_n}{\hbar^2}}\sqrt[4]{\frac{6E_n}{5kL^2}} = \pi + 2\pi n$$

Solving for E_n

$$E_n = \left(\frac{\pi}{2} + \pi n\right)^{\frac{4}{3}}\left(\frac{\hbar^2}{4mL^2}\right)^{\frac{2}{3}}\left(\frac{5kL^2}{6}\right)^{\frac{1}{3}}$$

Adding $E_{H,0}$ — zero energy of harmonic oscillators we get

$$E_n^{tot} = E_{H,0} + E_n$$

2 Monopole *(J. Parra Martinez)*

Consider a particle with mass m and charge e moving in a field of a magnetic monopole with magnetic charge g. Obtain the general classical non-relativistic trajectory of the particle and solve the scattering problem when the particle approaches the monopole with some initial velocity v_0, find the angle by which it would scatter. The impact parameter is fixed by the initial total angular momentum J of the system.

Solution:

The magnetic field of the monopole is given by

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3}$$

so the equations of motion are simply given by the Lorentz force

$$m\ddot{\mathbf{r}} = e\,\dot{\mathbf{r}} \times \mathbf{B} = \frac{eg}{r^3}\dot{\mathbf{r}} \times \mathbf{r}.$$

Let us check if the angular momentum of the particle is conserved on the trajectory

$$\frac{d}{dt}\left(m\mathbf{r} \times \dot{\mathbf{r}}\right) = m\mathbf{r} \times \ddot{\mathbf{r}} = \frac{eg}{r^3}\mathbf{r} \times (\dot{\mathbf{r}} \times \mathbf{r}) = \frac{eg}{r^3}(\dot{\mathbf{r}}\mathbf{r}^2 - \mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}})) = \frac{d}{dt}\left(eg\frac{\mathbf{r}}{r}\right).$$

So we find that the angular momentum of the particle is not conserved, but the total angular momentum, $\mathbf{J} = m\mathbf{r}\times\dot{\mathbf{r}} - eg\widehat{\mathbf{r}}$ is conserved. Note that this includes both the angular momentum of the particle $\mathbf{L} = m\mathbf{r}\times\dot{\mathbf{r}}$ and that of the electromagnetic field of the monopole and the charged particle $-eg\widehat{\mathbf{r}}$.

From this conservation law we get two pieces of information. First of all notice that $\widehat{\mathbf{r}}\cdot\mathbf{J} = -eg$, this means that the whole trajectory is contained in a cone whose axis is the angular momentum. Choosing polar coordinates with the z axis along $\mathbf{J}$ we find that the polar angle θ is fixed and its value is given by

$$\cos\theta_0 = \widehat{\mathbf{r}}\cdot\widehat{\mathbf{z}} = \widehat{\mathbf{r}}\cdot\frac{\mathbf{J}}{J} = -\frac{eg}{J} \tag{3}$$

where $J = |\mathbf{J}|$. Note that the sine and cosine of the angle are given respectively and up to an irrelevant sign, the fractions of the total angular momentum carried by the particle and the field.

Furthermore, we can also use the conservation of the absolute value J, which we can write in spherical coordinates as

$$J^2 = |\mathbf{L}|^2 + (eg)^2 = m^2r^4\sin^2\theta_0\dot{\phi}^2 + (eg)^2 \quad\rightarrow\quad L^2 = m^2r^4\sin^2\theta_0\dot{\phi}^2 = J^2 - (eg)^2$$

so we observe that even though $\mathbf{L}$ is not conserved, its modulus L does not change in time.

On the other hand, the monopole field is static, hence, since nothing depends explicitly on time, energy is conserved

$$E = \frac{m}{2}\dot{\mathbf{r}}^2 = \text{const.}$$

Notice that this implies that the absolute value of the velocity does not change:

$$E = \frac{m}{2}\dot{\mathbf{r}}^2 \quad\rightarrow\quad \dot{\mathbf{r}}^2 = \frac{2E}{m} = v_0^2.$$

Rewriting this in spherical coordinates and in terms of $\ell = L/m$ and v_0 we find that

$$E = \frac{m}{2}\left(\dot{r}^2 + \frac{\ell^2}{r^2}\right) = \frac{m}{2}v_0^2$$

which we can easily solve

$$\dot{r} = \pm\frac{1}{r}\sqrt{(v_0 r)^2 - \ell^2} \qquad \to t - t_0 = \pm\int \frac{dr\, r}{\sqrt{(v_0 r)^2 - \ell^2}} = \pm\frac{1}{v_0^2}\sqrt{(v_0 r)^2 - \ell^2}$$

yielding

$$r^2 = [v_0(t - t_0)]^2 + r_0^2$$

where t_0 and $r_0 = \frac{\ell}{v_0} = \frac{L}{m v_0}$ are the time and distances of closest approach respectively, in particular we can choose $t_0 = 0$. Next we can use this to solve for ϕ

$$\dot{\phi} = \frac{\ell}{\sin\theta_0\, r^2} \quad \to \quad \phi - \phi_0 = \frac{\ell}{\sin\theta_0}\int \frac{dt}{[v_0(t - t_0)]^2 + r_0^2} = \frac{1}{\sin\theta_0}\tan^{-1}\left(\frac{v_0 t}{r_0}\right)$$

again θ_0 correspondst to the azimuthal angle at closest approach ($t = t_0 = 0$) and we can choose our coordinates so that $\phi_0 = 0$. Substituting $\sin\theta_0 = \sqrt{1 - (eg/J)^2} = \frac{L}{J}$ the final solution for the dynamics of the particle takes form:

$$\begin{aligned} r^2 &= (v_0 t)^2 + r_0^2, \\ \theta &= -\cos^{-1}\left(\frac{eg}{J}\right), \\ \phi &= \frac{J}{L}\tan^{-1}\left(\frac{v_0 t}{r_0}\right). \end{aligned}$$

It is easy to describe what is going on: For $L = 0$, or equivalently $J = eg$, we find that $r_0 = 0$, $\phi = \pm\frac{J}{L}\frac{\pi}{2}$, so the particle travels in a straight line with fixed θ and ϕ and constant velocity until it reaches the monopole position. Then it bounces back in the same direction at the same velocity.

On the other hand, if $L \neq 0$, the particle comes in from infinity at $\phi(-\infty) = -\frac{\pi}{2}\frac{J}{L}$, it spirals down the cone $\cos\theta = -\frac{eg}{J}$ at constant velocity v_0 until it reaches $r(0) = r_0$ and $\phi(0) = 0$, and then it bounces back spiralling to infinity at $\phi(\infty) = \frac{\pi}{2}\frac{J}{L}$. Furthermore, from the equations above we can find the equation for the trajectory, first rewriting

$$\phi = \frac{J}{L}\tan^{-1}\left(\frac{v_0 t}{r_0}\right) \quad \to \quad r_0 \tan\left(\frac{L}{J}\phi\right) = v_0 t$$

and next using this in the r equation

$$r^2 = (v_0 t)^2 + r_0^2 = r_0^2\left(\tan^2\left(\frac{L}{J}\phi\right) + 1\right) = r_0^2 \sec^2\left(\frac{L}{J}\phi\right) \quad \to \quad r = \frac{r_0}{\cos\left(\frac{L}{J}\phi\right)}$$

From this equation we can observe that the amount of spiralling is controlled by the fraction of the total angular momentum carried by the particle, as we would have expected.

Now, the scattering angle in the equatorial plane is given by

$$\Delta\phi = \phi(+\infty) - \phi(-\infty) = \pi\frac{J}{L}$$

but we need the actual scattering angle, α_s, not just its projection onto a plane. However since the initial and final directions are known one can simply construct the unit vectors

$$\widehat{\mathbf{n}}_{\pm\infty} = \cos\theta_0(\sin\phi(\pm\infty)\widehat{\mathbf{x}} + \cos\phi(\pm\infty)\widehat{\mathbf{y}}) + \sin\theta_0\widehat{\mathbf{z}},$$

and calculate their dot product

$$\begin{aligned} \cos\alpha_s = \widehat{\mathbf{n}}_{+\infty}\cdot\widehat{\mathbf{n}}_{-\infty} &= \cos^2\theta_0(\sin\phi(+\infty)\sin\phi(-\infty) + \cos\phi(+\infty)\cos\phi(-\infty)) + \sin^2\theta_0 \\ &= \cos^2\theta_0\cos\Delta\phi + \sin^2\theta_0 = 1 - \cos^2\theta_0(1 - \cos\Delta\phi) = 1 - \frac{e^2 g^2}{J^2}\left(1 - \cos\left(\pi\frac{J}{L}\right)\right). \end{aligned}$$

Note that when the particle carries no angular momentum, the scattering angle is zero, which agrees with our discussion above.

3 Casimir Force *(V. Slepukhin)*

Calculate the Casimir force acting between two parallel perfectly conducting planes with massive bosonic field ($E = \sqrt{p^2 + m^2}$) inside and outside the planes.

Solution:

1 Introduction: massless and one-dimensional.

Let us begin with considering a pedagogical example of Casimir effect for massless particles in one dimension. Here we will provide a brief overview while more detailed explanation can be found in a variety of textbooks, for example, in [1].

Our system consists in a massless bosonic field in a one dimensional box. We know, that energy of any system in any state is given as a quantum average of the Hamiltonian in this state. Hence, the vacuum energy is $E = \langle 0|H|0\rangle$. If we write it in terms of operators of creation and annihilation, we get

$$E(a) = \langle 0|H|0\rangle = \sum_{n=1}^{\infty} \omega_n \langle 0|a^+ a + \frac{1}{2}|0\rangle = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n$$

For a massless field $\omega_n = k_n$. If size of the box is a, $k_n = \frac{\pi n}{a}$. Then we can substitute it to the previous equation

$$E(a) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\pi n}{a}$$

Here we obviously get a divergence and need to regularize it. You can find a lot of different regularization methods in [1]. Here we will use the simplest one, called zeta-function regularization. By definition, Riemann zeta-function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It is a well-defined function for $s > 1$. If we concider it as a function of a complex argument, we can analytically continue it to other regions. For example, one finds that $\zeta(-1) = -\frac{1}{12}$. Thus the energy can be rewritten as

$$E(a) = \frac{\pi\zeta(-1)}{2a} = -\frac{\pi}{24a}$$

and for the force we get

$$F = -\frac{d}{da}E(a) = -\frac{\pi}{24a^2}$$

2 One-dimensional massive.

For a massive field we have

$$\omega_n = \sqrt{p_n^2 + m^2} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + m^2}$$

and one can consider two cases, $m \ll \frac{1}{a}$, which in the limit of $m \to 0$ should reproduce the previous result, and the case $m \gg \frac{1}{a}$. Let's start with the last one. We know that the Casimir effect is tied with the infrared physics (large distances). Consequently, large n (carrying information on the ultraviolet sector) should give no contributions. We can start with the non-relativistic case decomposing the square root with small parameter $\frac{p}{m}$:

$$\omega_n(a) = \sum_{k=0}^{\infty} m C_k^{1/2} \left(\frac{\pi n}{am}\right)^{2k},$$

where $C_k^{1/2}$ is the combinatoric part of the Taylor expansion coefficient and then

$$E(a) = \sum_{k=0}^{\infty} \zeta(-2k) m \left(\frac{\pi}{am}\right)^{2k} C_k^{1/2}.$$

Note that here we used the definition of zeta-function. We know for $k \geq 1$ that $\zeta(-2k) = 0$ and $\zeta(0) = -1/2$. Then we get $E(a) = -m/2 = const$. Thus the force is zero in the non-relativistic limit or equivalently there is no Casimir effect for large m. This result is expected since the massive field can't propagate distances $l \gg \frac{1}{m}$.

Now consider the opposite case. If $\frac{1}{a} \gg m$, one can use $\frac{m}{p}$ as a small parameter:

$$\omega_n(a) = \sum_{k=0}^{\infty} m C_k^{1/2} \left(\frac{\pi n}{am}\right)^{1-2k}$$

and then

$$E(a) = \frac{1}{2} \sum_{k=0}^{\infty} \zeta(2k-1) m \left(\frac{\pi}{am}\right)^{1-2k} C_k^{1/2}.$$

The first term is the Casimir energy in the massless limit while the second term seems to be divergent since ζ has a pole at 1. To regularize this ambiguity let's assume that we have an extra plane in addition to two initial planes. Considering a plane in the between of two other planes we would be able to avoid dealing with the vacuum energy of the infinitely large volume which is a common source of such divergences. Thus we fix, say, the left plane and place the next one at the distance a to the right of the first one (we will calculate the force on this second plane) while the third plane is placed on the right from the system and fixed at a distance L from the first one. Here we assume $a \ll \frac{1}{m}$ and $(L-a) \ll \frac{1}{m}$. Now the energy of the middle plane is $E(a) = E_{12}(a) + E_{23}(L-a)$, where we marked planes by $1, 2, 3$, and for the force we have $F = -\frac{d}{da}E(a)$. Let's now consider just the divergent contribution ($k = 1$) to the energy

$$E^{(1)}(a) = E_{12}^{(1)}(a) + E_{23}^{(1)}(L-a) = \frac{1}{2}\zeta(1) m \frac{am}{\pi} C_1^{1/2} + \frac{1}{2}\zeta(1) m \frac{(L-a)m}{\pi} C_1^{1/2} = \frac{1}{2}\zeta(1) \frac{Lm^2}{\pi} C_1^{1/2}$$

and one readily finds that it doesn't depend on a anymore. That's it, this term gives zero contribution to the Casimir force and our regularizing procedure works. The final answer for the Casimir force caused just by the left plane is

$$F = -\frac{\pi}{24a^2} + \frac{3m^4 a^2 \zeta(3)}{16\pi^3} + O((am)^4)$$

Here we faced a common situation in QFT where some unphysical divergence appeared in the answer but could be moved to some infinite constant shift (here to the shift in the energy) which doesn't influence physical observable quantities, indeed, we are only interested in the energy difference not in its net value.

The next step in the understanding of the physics underlying this problem is to try to deal with the divergence by different methods and to compare results. For instance it could be more physical to consider the system of two planes in a finite but large box.

3 3D case.

Now consider a box $L \times L \times A$, and the second plane is in distance a parallel to one of the faces. For a massless field in a 3d box one finds

$$\omega_{nkl}(a) = \sqrt{p^2+m^2} = \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi k}{L}\right)^2 + \left(\frac{\pi l}{L}\right)^2 + m^2}$$

and the energy is

$$E(a) = \frac{1}{2} s \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi k}{L}\right)^2 + \left(\frac{\pi l}{L}\right)^2 + m^2}$$

here s correspond to number of possible polarizations of the particle (for scalar is 1).

Since L is large in comparison with all other parameters, we can partially change the summation by an integration:

$$E(a) = \frac{1}{2} s \sum_{n=1}^{\infty} \int_0^{\infty}\int_0^{\infty} dydz \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi y}{L}\right)^2 + \left(\frac{\pi z}{L}\right)^2 + m^2}$$

or, using the parity of function under the integral, we can go to the integral from minus to plus infinity:

$$E(a) = \frac{1}{8} s \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dydz \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi y}{L}\right)^2 + \left(\frac{\pi z}{L}\right)^2 + m^2}$$

The latter integral is divergent and to deal with it, we need a regularization. One could introduce a regularizing function $R(a, n, y, z)$ so that the energy converges

$$E(a) = \frac{1}{8} s \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dydz \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi y}{L}\right)^2 + \left(\frac{\pi z}{L}\right)^2 + m^2}\, R(a, n, y, z)$$

Let's now go to polar coordinates in (y, z)-plane assuming for simplicity that R doesn't depend on ϕ then

$$E(a) = \frac{1}{8} s\pi \sum_{n=1}^{\infty} \int_0^{\infty} \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi r}{L}\right)^2 + m^2}\, R(a, n, r) dr^2$$

or changing the variable to $x = \left(\frac{\pi r}{L}\right)^2$ we find for the integral

$$E(a) = \frac{L^2}{8\pi} s \sum_{n=1}^{\infty} \int_0^{\infty} \sqrt{\left(\frac{\pi n}{a}\right)^2 + x + m^2}\, R(a, n, x) dx$$

We can analyze the energy in a general form assuming that R depends only on $\frac{n}{a}$ and denoting the function under the integral as $f\left(\frac{n}{a}, x\right)$ and then we have

$$E(a) = \frac{sL^2}{8\pi} \sum_{n=1}^{\infty} \int_0^{\infty} f\left(\frac{n}{a}, x\right) dx$$

while for the rest of the box

$$E(A-a) = \frac{sL^2}{8\pi} \sum_{n=1}^{\infty} \int_0^{\infty} f\left(\frac{n}{A-a}, x\right) dx$$

Noting that $A-a$ is large and substituting $y = \frac{n}{L-a}$ one can transform the sum into an integral

$$E(A-a) = (A-a)\frac{sL^2}{8\pi}\int_0^{\infty}\int_0^{\infty} f(y,x)dydx = \frac{sL^2 A}{8\pi}\int_0^{\infty}\int_0^{\infty} f(y,x)dydx - \frac{asL^2}{8\pi}\int_0^{\infty}\int_0^{\infty} f(y,x)dydx$$

The first term doesn't depend on a at all so after taking the derivative it will go away.

The total energy (part, that can give a contribution to the force) is

$$E(a) + E(A-a) = \frac{sL^2}{8\pi}\left(\sum_{n=1}^{\infty}\int_0^{\infty} f\left(\frac{n}{a}, x\right)dx - \int_0^{\infty}\int_0^{\infty} f\left(\frac{n}{a}, x\right)dndx\right)$$

where we changed the integration variable by $y = \frac{n}{a}$. Now we can use Euler-Maclaurin formula to evaluate the difference between the sum and the integral

$$E(a) + E(A-a) = -\frac{sL^2}{8\pi}\sum_{n=1}^{\infty} B_n \frac{F^{(n-1)}(0) - F^{(n-1)}(\infty)}{n!}$$

where $F(n) = \int_0^{\infty} f\left(\frac{n}{a}, x\right)dx$ and B_n is the nth Bernoulli number. We want our regulator function to go to zero at infinity fast enough, so we just through away $F^{(n-1)}(\infty)$. Then we have

$$E(a) + E(A-a) = -\frac{sL^2}{8\pi}\sum_{n=1}^{\infty} B_n \frac{F^{(n-1)}(0)}{n!}$$

Also we want the regulator to be very slow and equal to unit near zero. Than we have

$$E(a) + E(A-a) = -\frac{sL^2}{8\pi}\sum_{k=1}^{\infty} B_k \frac{\int_0^{\infty} f(0,x)^{(k-1)}dx}{k!}$$

where the derivative is by n.

Changing back from n/a to y one finds

$$E(a) + E(A-a) = -\frac{sL^2}{8\pi}\sum_{k=1}^{\infty} a^k B_k \frac{\int_0^{\infty}(\frac{d}{dy})^{k-1} f(0,x)dx}{k!}$$

where we know that $f(y,x) = \sqrt{y^2 + x + m^2}R(y,x)$ and thus all odd derivatives from this function are zero since $R(y,x)$ is a slow function of y and could be considered as a constant at $y = 0$ (Note that all odd B_n are zero too). This result is unphysical since if the mass is very small we should return to the massless case result. However one has to expect that since the series does not converge and the theorem is not applicable. (Homework: understand explicitly, what is wrong here).

To deal with this issue we can consider two limits: $m \gg 1/a$ and $m \ll 1/a$. For the first one $f(y,x) = m\sum_{n=1}^{\infty} C_n^{1/2}(y^2+x)^n R(y,x)$. Here all odd derivatives at $y = 0$ are also zero but it is consistent since in the limit of heavy particle there should be no force. However the limit of light mass requires further investigation and we have to return to the formula for the energy

$$E(a) = \frac{sL^2}{8\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\sqrt{\left(\frac{\pi n}{a}\right)^2 + x + m^2}R\left(\left(\frac{\pi n}{a}\right)^2 + m^2 + x\right)dx$$

or shifting the integration variable

$$E(a) + E(A-a)) = \frac{sL^2}{8\pi}\left(\sum_{n=1}^{\infty}\int_{\left(\frac{\pi n}{a}\right)^2+m^2}^{\infty}\sqrt{x}R(x)dx + \sum_{n=1}^{\infty}\int_{\left(\frac{\pi n}{A-a}\right)^2+m^2}^{\infty}\sqrt{x}R(x)\right)$$

Taking the derivative with respect to a to get force

$$-F = \frac{d}{da}(E(a) + E(A-a)) = \frac{sL^2}{4\pi}\sum_{n=1}^{\infty}\sqrt{\left(\frac{\pi n}{a}\right)^2 + m^2}\frac{\pi^2 n^2}{a^3}R\left(\left(\frac{\pi n}{a}\right)^2 + m^2\right)$$
$$-\frac{sL^2}{4\pi}\sum_{n=1}^{\infty}\sqrt{\left(\frac{\pi n}{A-a}\right)^2 + m^2}\frac{\pi^2 n^2}{a^3}R\left(\left(\frac{\pi n}{a}\right)^2 + m^2 +\right)$$

and using the fact that $m \ll \frac{1}{a}$ we get

$$\frac{d}{da}E(a) = \frac{sL^2}{4\pi} \sum_{n=1,k=0}^{\infty} C_k^{1/2} \left(\frac{am}{\pi n}\right)^{2k} \frac{\pi^3 n^3}{a^4} R\left(\left(\frac{\pi n}{a}\right)^2 + m^2\right) = \frac{sL^2}{4\pi^2} \sum_{n=1,k=0}^{\infty} C_k^{1/2} \left(\frac{a^{k-4} m^k}{\pi^{k-4} n^{k-3}}\right) R\left(\left(\frac{\pi n}{a}\right)^2 + m^2\right)$$

or after the zeta-function regularization

$$\frac{d}{da}E(a) = \frac{sL^2}{4\pi^2} \sum_{k=0}^{\infty} C_k^{1/2} \left(\frac{am}{\pi}\right)^{2k-4} \zeta(2k-3) m^4$$

There is a singular term at $k = 4$ but fortunately this term could be dropped exactly in the same way as for the one dimensional case (then we will assume $m \ll \frac{1}{A}$) . Thus the final answer reads

$$\frac{d}{da}E(a) = \frac{sL^2\pi^2}{480a^4} - \frac{sL^2m^2}{96a^2} + O(m^3)$$

or for the force between two planes (without box)

$$F = -\frac{sL^2\pi^2}{480a^4} + \frac{sL^2m^2}{96a^2} + O(m^3)$$

4 Radiation Reaction Force *(A. Sadofyev)*

Consider the theory of massive electrodynamics with the action

$$S = \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu \right) d^4 x \,.$$

Suppose there is an external force acting on a point-like charge resulting in its motion along some general trajectory. Calculate the back-reaction force caused by the field of the charge, work out the example of circular motion. Reproduce the usual electrodynamics answer for the radiation reaction force by the massless photon limit. Repeat the same consideration for the force in terms of quantum field theory and compare your results.

Solution:

For simplicity we could start with classical scalar electrodynamics. This problem was widely considered in the literature (see e.g. [2] and we will use their notations). The appropriate action is

$$S = -m \int \sqrt{\dot{z}^2} d\tau + \int d^4 x j(x) \phi(x) + \int \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) d^4 x \,,$$

where

$$j(x) = \int \sqrt{\dot{z}^2} \delta^{(4)}(x - z(\tau)) d\tau$$

here $z^\mu(\tau)$ is the trajectory and τ is chosen that $\dot{z}^2 = 1$.

For EOMs one immediately finds

$$\begin{aligned} m\ddot{z}^\mu &= -\left(\partial^\mu - \dot{z}^\mu \dot{z}^\nu \partial_\nu - \ddot{z}^\mu\right) \phi(z) \\ \left(\partial^2 + m^2\right) \phi(x) &= j(x) \,. \end{aligned}$$

We firstly should analyze how a source produces scalar field around it. To do so one has to consider the Green function for the d'Alambert equation

$$\left(\partial^2 + m^2\right) G(x, x') = \delta^{(4)}(x - x') \,.$$

We also should concentrate on the retarded Green function since there are no signals with speed above the speed of light (note that $c = 1$ here) and using Lorentz symmetry we can deduce that $G_R(x, x') = \theta(x^0 - x'^0) G(\sigma)$ with $\sigma = (x^0 - x'^0)^2 - (\vec{x} - \vec{x}')^2$ and for the field produced by the source we have $\phi_{self} = \int G_R(x, x') j(x') d^4 x'$.

Now it is time to turn to the force calculation and we note that

$$\begin{aligned} m\ddot{z}^\mu &= F^\mu_{ext} + F^\mu_{self} \\ m\ddot{z}^\mu &= -\left(\partial^\mu - \dot{z}^\mu \dot{z}^\nu \partial_\nu - \ddot{z}^\mu\right) \phi(z) \end{aligned}$$

where we also suppose existence of some external force making our source to move along its trajectory (this role could be played by some distribution of an external field). Thus one finds that

$$F^\mu_{self} = -\left(\partial^\mu - \dot{z}^\mu \dot{z}^\nu \partial_\nu - \ddot{z}^\mu\right) \int G_R(x, x') j(x') d^4 x' \,.$$

One could rewrite that as

$$F^\mu_{self} = -\left(\partial^\mu - \dot{z}^\mu \dot{z}^\nu \partial_\nu - \ddot{z}^\mu\right) \int_{-\infty}^{\tau(x^0)} G_R(\sigma) d\tau'$$
$$F^\mu_{self} = -\int_0^\infty \left(2P^\mu_\nu y^\nu \frac{d}{d\sigma} - \ddot{z}^\mu\right) G_R(\sigma) ds$$
$$F^\mu_{self} = \int_0^\infty \left(2P^\mu_\nu \frac{d}{d\sigma}\left(y^\nu \frac{ds}{d\sigma}\right) + \ddot{z}^\mu \frac{ds}{d\sigma}\right) G_R(\sigma) d\sigma\,,$$

where τ is a solution of $z^0(\tau) = x^0$, $y = z(\tau) - z(\tau')$, $P_{\mu\nu} = \delta_{\mu\nu} - \dot{z}_\mu(\tau)\dot{z}_\nu(\tau)$, $s = \tau - \tau'$ and $\sigma = (z(\tau) - z(\tau'))^2$.

Now let's start with the massless limit where the Green function reads

$$G_R(\sigma) = \frac{1}{2\pi}\theta(x^0 - x'^0)\delta(\sigma)$$

and thus we could plug $\sigma = 0$ under the integral in the self-force. Note that the retarded Green function of the massless field is non-zero only on the light cone unlike for the massive field. The reason for this is the following: the retarded Green function between points x and 0 ($x > 0$) is determined by the particles "emitted" from point 0 and "detected" at point x. Massless particles all propagate with the speed of light, regardless of the momenta, and thus all the "detected" particles should be on the same light cone as x. For the massive particles, there is a spectrum of velocities since $\omega = \sqrt{p^2 + M^2}$. Thus the retarded two-point function is non-zero for any pair of points which are connected by a time-like interval, since for any point of "detection" there are arriving slower particles which have been emitted earlier and faster ones emitted later.

Now, back to the self-force, one can expand $2P^\mu_\nu \frac{d}{d\sigma}\left(y^\nu \frac{ds}{d\sigma}\right)$ and $\frac{ds}{d\sigma}$ in powers of σ:

$$\frac{ds}{d\sigma} = \frac{1}{2\sqrt{\sigma}} + \frac{\ddot{z}^2}{8}\sqrt{\sigma} + \ldots \cdot \sigma^{\frac{3}{2}}$$
$$y^\mu = \dot{z}^\mu \sqrt{\sigma} - \ddot{z}^\mu \frac{\sigma}{2} + \left(\dddot{z}^\mu + \frac{\dot{z}^\mu \ddot{z}^2}{4}\right)\frac{\sigma^{\frac{3}{2}}}{6} + \ldots \cdot \sigma^2$$

to obtain the first expansion one should consider expansion of $\sigma = y^2$ and then inverse it for $s(\sigma)$. Thus the answer for the self force reads

$$F^\mu_{self} = \int_0^\infty \left[\frac{\ddot{z}^\mu}{4\sqrt{\sigma}} + \frac{1}{6}\left(\dddot{z}^\mu + \dot{z}^\mu \ddot{z}^2\right) + \ldots \cdot \sqrt{\sigma}\right] G_R(\sigma) d\sigma\,.$$

Note that the retarded theta function is absorbed into the integration interval and could be dropped. Substituting the Green function for the massless case one finds that the term proportional $\ddot{z}^\mu$ is divergent. This is a common problem of classical electrodynamics and for now we would use a trick absorbing this divergence into redefined mass (so called renormalized). Finally one gets the well known result

$$m_{ren}\ddot{z}^\mu = F^\mu_{ext} + \frac{1}{12\pi}\left(\dddot{z}^\mu + \dot{z}^\mu \ddot{z}^2\right)\,.$$

We could now turn to the massive case and to do so let's consider the appropriate Green function

$$G_R(\sigma) = \frac{1}{2\pi}\theta(x^0 - x'^0)\left(\delta(\sigma) - \theta(\sigma)\frac{mJ_1(M\sqrt{\sigma})}{2\sqrt{\sigma}}\right)$$

which could be found in the literature or by a direct calculation. However it is known that $J_1(x) \sim 1/\sqrt{x}$ for $x \ll 1$ and our expansion doesn't work anymore (indeed, $\int d\sigma \sigma^n J_1(m\sqrt{\sigma})$

diverges for $n > 1/2$). However one could check that the massive contribution to the force in the non-expanded form is finite and to proceed further we have to restrict ourselves to a specific trajectory avoiding extra algebra.

In this problem we are asked to consider a circular motion and one finds the trajectory to be

$$z^\mu = (\gamma\tau, \rho\cos\gamma\omega\tau, \rho\sin\gamma\omega\tau, 0)\,,$$

where $\gamma = (1-\rho^2\omega^2)^{-\frac{1}{2}}$ and one can check that $\dot{z}^2 = 1$. Separating the mass renormalization and the self force

$$F^\mu_{self} = -F_m\ddot{z}^\mu + F_{self}(\dddot{z}^\mu + \dot{z}^\mu\ddot{z}^2)\,.$$

after some algebra skipped here for brevity we finally get

$$F_{self} = -\int_0^\infty ds\rho\frac{\gamma\omega s\cos\gamma\omega s - \sin\gamma\omega s}{(\gamma s - \rho^2\omega\sin\gamma\omega s)^2}\frac{mJ_1(m\sqrt{\sigma})}{4\pi\sqrt{\sigma}} - \frac{\rho\gamma^5\omega^3}{12\pi}\,.$$

where the last term is the massless limit while the first one is the field mass correction.

Thus the question of the problem is partially answered and we obtained a tool to proceed to the discussion of EM field case. However we leave the rest of the problem open for the interested reader who will have to generalize this consideration to the case of EM field. It would be also instructive to try to analyze the problem in terms of QFT and to compare the results.

5 Astronaut on a chain *(A. Shtyk)*

Consider a space station of mass M on the orbit of radius R around a planet of mass $M_0 \gg M$. In open space near the station there is an astronaut of mass $m \ll M$. The astronaut is attached to the station via the chain that consists of $N \gg 1$ links each of length a. Links can be considered as rigid rods that are connected at their ends and can freely rotate one around another. The chain is reasonably short with the total length $Na \ll R$. Assuming that microwave background radiation keeps the system at temperature T, find the distance from the astronaut to the station.

Solution:

The key aspect of the problem is a presence of a *phase transition* that happens for a certain temperature T_c. In this phase transition the coordinate of the astronaut x plays the role of an *order parameter*. In high-temperature phase the heated chain manages to keep the astronaut near the station with average position of the astronaut $\langle x \rangle = 0$, while in low-energy phase $T < T_c$ a non-zero order parameter $\langle x \rangle \neq 0$ develops.

Here we focus on the mean-field analysis of the phase transition in the system with the main goal to identify T_c.

Effective potential energy of the astronaut

It is convenient to use a non-inertial reference frame rotating with a space station, where the effective potential energy of an astronaut could be written as

$$U_0 = -\gamma \frac{mM_\oplus}{r} - \frac{1}{2} m\omega^2 r^2 = -m\omega^2 R^2 \left(\frac{R}{r} + \frac{1}{2} \frac{r^2}{R^2} \right), \tag{4}$$

where r is the radius coordinate of the astronaut and ω is an angular frequency of the space station's orbital motion. Expanding the energy near the station's orbit in powers of $x = r - R$ we have

$$U_0(x) = \text{const} - \frac{3}{2} m\omega^2 x^2 + \mathcal{O}(x^3). \tag{5}$$

This leads to unstable equilibrium at $x = 0$ with a force

$$F_0 = -\partial_x U_0 = +3m\omega^2 x. \tag{6}$$

Free energy of the chain

The force F_0 at non-zero temperature competes with the entropic force of the chain. In order to find the latter let us imagine a fixed force F applied to the ends of the chain. For such a setup a partition function of the chain is

$$Z_{\text{chain}} = \int \prod_i (\mathrm{d}\Omega_i) \exp\left[-\frac{F \sum_i a \cos\theta_i}{T} \right] = \left(\int_0^\pi \frac{\sin\theta \mathrm{d}\theta}{2} \mathrm{e}^{-(Fa/T)\cos\theta} \right)^N = \left(\frac{\mathrm{sh}(Fa/T)}{(Fa/T)} \right)^N. \tag{7}$$

This gives chain's free energy

$$\mathcal{F}_{\text{chain}} = -NT \ln Z_{\text{chain}} = -NT \ln \frac{\mathrm{sh}(Fa/T)}{(Fa/T)}, \tag{8}$$

The length of the chain is thus given by

$$x = -\frac{\partial \mathcal{F}_{\text{chain}}}{\partial F} = Na \left(\coth \frac{Fa}{T} - \frac{T}{Fa} \right). \tag{9}$$

where by x we mean the average over the thermal fluctuations $\langle x \rangle_T$.

Self-consistency equation

The position of the astronaut within the mean-field approximation is given by the self-consistency equation that results from (9) plus the fact that the actual force must in fact be $F_0 = -\partial_x U_0$,

$$x = Na\left(\coth\frac{3m\omega^2 ax}{T} - \frac{T}{3m\omega^2 ax}\right), \tag{10}$$

or equivalently

$$\boxed{\xi = \coth\frac{3\xi}{\tau} - \frac{\tau}{3\xi},} \tag{11}$$

where $\xi = x/Na, \tau = T/T_c$ and $T_c = m\omega^2(Na^2)$ is the transition temperature. The *mean-field* solution in limiting cases is thus

$$x = Na \times \begin{cases} 0 & T > T_c \\ \left(\frac{5}{3}\frac{T_c - T}{T_c}\right)^{1/2} & T \lesssim T_c \\ \left(1 - \frac{T}{3T_c}\right) & T \ll T_c \end{cases}. \tag{12}$$

Note that for $T < T_c$ there is also a second solution that differs only by sign, so that there are two equilibria for the astronaut, above and below the station orbit. Say, at zero temperature we have $\pm Na$.

It is interesting to estimate the critical temperature for realistic parameters. For example for the astronaut of mass 80 kg on a 10 meter chain with 100 segments near the ISS (orbital period 90 minutes) we would get

$$T_c \sim 10^{19}\ \text{K} \sim 1000\ \text{TeV!} \tag{13}$$

This energy scale is beyond even current LHC energies ($\sim$ 10 TeV). The problem has little practical relevance in a given context but the situation is very different for proteins in a centrifuge for example.

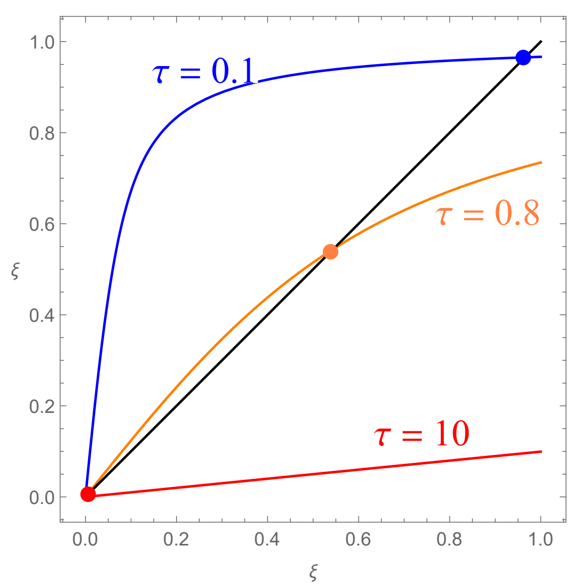

Figure 2: Graphical representation of the self-consistency equation. The black line is $y = \xi$, while blue, orange and red give RHS of (11) for three different temperatures τ.

6 Landau Levels in a Box *(A. Sadofyev)*

Obtain the energy spectrum of relativistic fermions of mass m placed in external constant magnetic field B (along z-axis). Suppose the system to be of a finite size l along x-direction while y, z-directions are not confined.

Solution:

This problem could be considered to be requiring some minor scientific investigation through the literature. Since the problem was mainly considered by participants in a too simplified form we provide only a set of references to let to an interested reader to find the final solution. The main non-trivial issue with a relativistic fermion in a box consists in the fact that one implying usual zero wave function boundary conditions would get a trivial solution in the whole region (however there are more issues here, say the mixing with negative energy solutions). As a first step towards the answer we suggest here a short consideration of a relativistic fermion in a 1d box along some recent review [3] (for further references look at [4]). The general case in the presence of an external magnetic field could be obtained by some generalization which requires further study.

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