# International Theoretical Physics Olympiad

January, 2021

You have 24 hours to solve the problems below. You can use any textbook and online materials published before the beginning of the contest, provided that you cite your references appropriately. You can also use Computer Algebra Systems like Mathematica or Maple. However, in case you want to present a solution that relies on these computations, please provide a complete description and all supplementary materials, including the source code. You can upload your solutions using your team page, or e-mail them to us at **info@thworldcup.com** if there are any technical difficulties. Although we can accept almost all possible formats, .pdf is preferred. Please submit solutions to only one account per team. Good luck!

**Exercise (*1*): Melanoheliophobic mouse.**

Eine Maus, Siegfried, is stuck in a labyrinth. Siegfried was a house mouse of Albert Einstein, so it knows a bit of General Relativity. Because of that, Siegfried is afraid of black holes. Please help Siegfried overcome this fear, by calculating the probability that the labyrinth collapses into a black hole. Siegfried is smaller than we are, and can see smaller things than us, like extra dimensions. Hence, assume that Siegfried lives in $d$-dimensions. The labyrinth could be considered as a graph where each vertex has a valence $q$. The labyrinth is random, meaning that edges coming out of each vertex are uniformly distributed on an $S^{d-1}$-sphere. Each edge is weightless and has length $L$, and each vertex has mass $m$. What would you say about the same labyrinth, when the mass is uniformly distributed along each edge while the vertices are weightless?

**Exercise (*2*): A different kind of virus.**

Dangerous bacterial strains are able to adapt to antibiotic treatments by gene manipulation. Rather than sexual or asexual reproduction, which both involve the creation of new bacterial units to achieve genetic mutations, some bacteria are able to "infect" other living bacteria with foreign genetic code. Viral transfer of DNA is hypothesized to be a dominant contributor to the spread of antibiotic resistance.

Imagine lining a square petri dish (with side length $L$) with a linearly increasing concentration of an antibiotic (with slope $\kappa$), and a uniform distribution of food. Place an initial bacterial population on one boundary of the dish, where the antibiotic concentration is 0. Then, model the dynamics of the population, and calculate (either analytically or numerically):

1) The time it takes for the bacteria to reach the opposite boundary of the dish where the antibiotic is maximally concentrated.

2) The percentage of resistant bacteria in the population over time, and the distribution.

Assume the bacteria are identical, except for a small fraction of the initial population $\epsilon$ which has a mutation that makes it partially resistant to the antibiotic. In the absence of the antibiotic, the rate of reproduction of the resistant bacteria is slower by a small amount $\delta$, as the developed resistance has a metabolic cost. You may assume that the bacteria have a characteristic size $\ell \ll L$, that the bacteria do not move or overlap, and that a single resistant bacterium will infect its nearest neighbors with the resistance gene at a constant rate $r$. You will have to reason about the effect of the antibiotic on the rate of reproduction of the bacteria. With justification, you may assume that the decrease in the rate of reproduction is linear in the total amount of antibiotic in contact with a bacterium, with differing slopes $c_{nr}$ and $c_r$ between the populations (where $c_r < c_{nr}$).

**Exercise (*3*): Cut to the trace.**

Consider the following matrix path integral:

$$\mathcal{Z} = \int \prod_{i,j} dM_{ij}\, e^{-\frac{N}{2}\mathrm{Tr}\,(M^2)+N^2 V(M)} \tag{1}$$

where the integration is over the elements of an $N \times N$ Hermitian matrix and $V(M)$ is an arbitrary potential. Such path integrals are understood to be dual to *minimal string theories*, whose worldsheet actions involve Liouville gravity with a minimal conformal field theory in the matter sector.

1) Take $V(M) = \frac{h}{N^2}\left(\mathrm{Tr}\, M\right)^2$, where $h$ is a coupling constant. Calculate the probability distribution of a single eigenvalue of $M$, assuming $N$ is very large.

2) Compute the same distribution assuming $V(M) = \frac{g}{N}\mathrm{Tr}\, M^{2k} + h\left(\exp\left(\frac{1}{N}\mathrm{Tr}\, M^{2k}\right) - 1\right)$. (You will receive partial credit if you solve for specific values of $k$).

**Exercise (*4*): The fabric of spacetime.**

There is a common illustration of how gravitational forces arise within the theory of general relativity (GR). Consider a trampoline, a stretched piece of elastic fabric, with a heavy ball placed in its center. If smaller balls are placed on the fabric of this toy universe, they will move along trajectories resembling ones in the gravitational field of a heavy object. In this problem you are suggested to study the system described above and answer the following questions:

0) Imagine that a coordinate grid was drawn on the flat trampoline. Before being deformed by the heavy ball, the metric was flat. Describe the metric after the deformation. Does it look like a Schwarzschild solution?

1) What are the possible trajectories of a ball on the trampoline deformed by a heavy ball at the center? Assume that the balls have finite radii, that there is friction, and that you may work in the probe limit for now.

2) What are the leading effects of the boundary shape on the ball trajectories? Assume that the boundary of the trampoline can be drawn on a flat piece of paper.

3) In GR, two orbiting objects will radiate gravitational waves and eventually collide. Is there an analogous radiation emitted by balls moving on the trampoline and propagating in its fabric? If so, estimate the intensity of this radiation.

**Exercise (5): The Pursuit of Maximal Happyness.**

The animal world of the Flatland (a reference to the novel by E.A. Abbott about a 2D country) is quite versatile. Take, for example, the myriapods. These creatures consist of a body with $N$ segments, $N \gg 1$. Each segment is a piece of a straight line of length $\delta$. In the middle of each segment there is one leg, a straight interval of length $\frac{a}{2}$, such that $N\delta \gg \frac{a}{2} \gg \delta$. Legs are always orthogonal to the segments and can go from one side of the segment to another if needed. In Flatland, myriapods can pass through each other like ghosts.

Myriapods prefer to sit straight. Their happiness level depends on the angle $\theta_{i,i+1}$ between the consecutive segments as $\sum_i \kappa \cos\theta_{i,i+1}/\delta$. They are also very extroverted animals. Namely, they like to hold each others' legs so that the ends of their legs coincide, forming a straight piece of length $a$. Their happiness level has a contribution proportional to the number of held legs with constant $\mu\delta$. Assume $N\delta \gg \sqrt{\frac{\kappa}{\mu}} \gg \delta$.

Consider a system of two myriapods. Obviously, the state of maximal happiness is when both myriapods are perfectly straight and hold all legs. However, due to mistakes upon their meeting they may end up in a metastable state, with the happiness level lower than a maximal possible, but such that the transition from this state to the state of maximal happiness would require an initial decrease in happiness by an amount proportional to the total number of segments.

1) Classify the possible metastable states (defects). When analyzing defects which are observable even when $a = 0$, you may simplify and set $a = 0$. Otherwise, keep $a$ nonzero.

2) Consider pure defects of each type, such that effects of other defect types are not present. How many independent parameters does each defect have? Do these parameters completely determine the shape of the myriapods?

3) In each class of defect, determine the angle between directions of the myriapods at infinity, both on the left and on the right of the defect, as a function of the defect parameters, as well as $\kappa$ and $\mu$.

4) One class of defects demonstrates a phase transition when parameters of the problem are varied while the parameters of the defects are fixed. Identify this class, as well as the behavior in the vicinity of the phase transition. You may solve this part of the problem numerically.

**Exercise (*6*): Paper pushers.**

Most dark matter direct detection experiments involve massive collaboration efforts surrounding large-scale engineering projects and cutting edge instrumentation. Your challenge is to design a dark matter detector using much more rudimentary materials, but unbound by real-world financial or practical constraints.

1) You might remember childhood activities in which you were given paper, scissors, glue, and creative freedom. Perhaps you even yearn to relive those times. Well, good news! Your government has an infinite surplus of paper and glue, and has passed it off to you, free of charge. Use these resources to construct a detector of self-interacting dark matter. Because you have saved so much money on the apparatus itself, your detector is not required to be on Earth, or anchored to any object for that matter. You may also assume convenient qualities for your "paper" and the environment of your detector, to modify aspects such as density, combustion point, temperature, etc. within reasonability.

2) Now that you have a potential method of detection, what physical characteristics of your detector can be optimized? Is there an ideal volume or density of your detector which will maximize the rate of detected events?