1 Quantum to Classical

(idea and solution by A. Phan)

Even start out initially with quantum fuzziness, a macroscopic object will eventually becomes classical in a very short time due to the loss of quantum entanglement through interactions with the environment. Obtain this time scale, given the mass of the environment particles m (very small compare to that of the macroscopic object), the collision rate Γ between the object and the environment particles, the thermal equilibrium temperature T of the environment and the spatial resolution L.

Solution:

This solution is based on the work [1].

Consider the a local interaction between a generic system of mass M with an environment particle of mass m:

$$M \gg m \tag{1}$$

Locality in space and time means the position of the system X and the position of the enviroment particle x remain the same through out the collision. For simplicity, look at the physics of 1 spatial dimension. Also, the environment particle move much faster than the system, so that initially, the system has momentum P and the particle has momentum p:

$$\frac{P}{M} \ll \frac{p}{m} \tag{2}$$

Right after the collision with energy and momentum conservation the results can be approximated to be:

$$P' \approx P + 2p \tag{3}$$

$$p' \approx -p \tag{4}$$

Initially, the whole **system** has Schrodinger wavefunction $\tilde{\Psi}(P, p)$ in momentum space, hence in position space:

$$\Psi(X, x) = \int \frac{dPdp}{2\pi} \tilde{\Psi}(P, p) e^{iPX + ipx} \tag{5}$$

Right after the collision, the Schrodinger wavefunction becomes:

$$\Psi'(X, x) = \int \frac{dPdp}{2\pi} \tilde{\Psi}(P, p) e^{iP'X + ip'x} \tag{6}$$

Note that in the regime of interests, the unchanging position but shifting momentum can also be seen as the shifting postion but unchanging momentum:

$$X' = X \quad , \quad x' = 2X - x \quad \Rightarrow \quad P'X + p'x \approx PX' + px' \tag{7}$$

$$\Psi'(X, x) = \int \frac{dPdp}{2\pi} \tilde{\Psi}(P, p) e^{iPX' + ipx'} = \Psi(X', x') \tag{8}$$

The quantum information can be encoded more conveniently inside the density matrix to deal with decoherence due to coarse-graining out the environment:

$$\mathbb{R}(X, x; Y, y) = R(X, Y) r(x, y) \quad \text{(before)} \tag{9}$$

$$\mathbb{R}'(X, x; Y, y) = \mathbb{R}(X', x'; Y', y') = R(X', Y') r(x', y') = R(X, Y) r(x', y') \quad \text{(after)} \tag{10}$$

To get the density matrix of the system after the collision, do the partial trace over all the quantum state of the enviroment particle:

$$\Lambda = \text{Tr}\left[r(x', y')\right] = \int dz r\big(z + 2(X - Y), z\big) \tag{11}$$

$$R'(X,Y) = R(X,Y)\Lambda \tag{12}$$

For a laboratory experiment the setting is always spatially localized, therefore the density matrix and its derivatives becomes extremely small for large separation $|X - Y|$, hence one can just go use the Taylor expansion up to 2nd order. Using the momentum operator in spatial position basis $p = -i\partial_z$:

$$\Lambda \approx \int dz r(z,z) + 2(x-y)\int dz \partial_w r(w,z)\Big|_{w=z} + 2(x-y)^2 \int dz \partial_w^2 r(w,z)\Big|_{w=z} \tag{13}$$

$$= \langle 1 \rangle + 2i(x-y)\langle p \rangle - 2(x-y)^2 \langle p^2 \rangle \quad , \quad \langle ... \rangle = \mathrm{Tr}(r...) \tag{14}$$

To consider the collective behavior of environmental degrees of freedom sucking out quantum information, let's look at many environment particles in thermal equilibrium temperature T and the collision rate between the system and the particles Γ:

$$\langle p \rangle = 0 \quad , \quad \langle p^2 \rangle = mkT \quad , \quad \delta R = \Gamma \delta t \big(\Lambda - \langle 1 \rangle\big) R \tag{15}$$

The contribution to the equation of motion for the system density matrix in the time continuum limit:

$$\partial_t R(X,Y) \supset -2\Gamma mkT(X-Y)^2 R(X,Y) \tag{16}$$

In full, with the intrinsic quantum dynamics, one arrives at the quantum Brownian equation:

$$\partial_t R(X,Y) = \frac{i}{2M}\big(\partial_X^2 - \partial_Y^2\big) R(X,Y) - 2(X-Y)^2 \Gamma mkT R(X,Y) \tag{17}$$

The off-diagonal $X \neq Y$ part of the density matrix dies out with time, hence leads to the coherence – decoherence transition as the physics start from quantum regime goes toward classical regime. In other words, the density matrix evolves to be only non-zero in (X, X) components – diagonalized in the spatial position basis (hence classical density matrix), which means position localization is the behavior of the object in the classical world. Given the spatial resolution of the physical observation to be L, one can estimate the decoherence time:

$$\tau \sim \frac{1}{\Gamma L^2 mkT} \tag{18}$$

References

[1] Gamble, John King, and John F. Lindner. "Demystifying decoherence and the master equation of quantum Brownian motion." *American Journal of Physics* 77.3 (2009): 244-252.