2 Square Plate

(solution F. Popov)

A 2d metallic square $\mathcal{K} = [0, a] \times [0, a]$ has a given resistivity density tensor:

$$\hat{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \quad , \quad \vec{E} = \hat{\rho}\vec{j}\,.$$

There is a potential difference between two opposite sides of the square. In this problem you're asked to calculate the full current and the resistivity.

Solution:

Let us consider the following problem. We are given a conducting square with a strange conductivity tensor

$$\vec{E} = \hat{\rho}\vec{j}, \quad \hat{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \tag{1}$$

We apply the voltage to the opposite sides of the square $\Delta\phi = \phi_1 - \phi_0 = V > 0$. We need to find a current and resistivity of the given system $\mathbb{K} = [0, a] \times [0, a]$. Let us notice that electric field and current satisfy the following equations

$$\operatorname{div} E = \operatorname{rot} E = 0, \quad \operatorname{div} j = \operatorname{rot} j = 0 \tag{2}$$

But the current and electric field satisfy different boundary conditions

$$E_x|_{y=0,a} = 0,\, (\rho_{xx}E_x - \rho_{xy}E_y)|_{x=0,a} = 0, \quad j_x|_{x=0,a} = 0,\, (\rho_{xx}j_x + \rho_{xy}j_y)|_{y=0,a} = 0 \tag{3}$$

One can notice that under the action of mirror symmetry $x' = y, y' = x$ vector transfors as $j'_y = j_y, j'_x = -j_x$ and still satisfies the equations $\operatorname{div} j = \operatorname{rot} j = 0$, but boundary conditions change

$$j'_x|_{y=0,a} = 0,\, \left(\rho_{xx}j'_x - \rho_{xy}j'_y\right)\Big|_{x=0,a} = 0 \tag{4}$$

Therefore we can establish the relation between E and j. It is easy to see that$\vec{E} = \sqrt{\rho_{xx}^2 + \rho_{xy}^2}\vec{j}'$. Then the resistivity is

$$R = \frac{\int\limits_0^a E_y(0, y)dy}{\int\limits_0^a j_y(x, 0)dx} = \sqrt{\rho_{xx}^2 + \rho_{xy}^2}\frac{\int\limits_0^a j'_y(0, y)dy}{\int\limits_0^a j_y(x, 0)dx} = \sqrt{\rho_{xx}^2 + \rho_{xy}^2} \tag{5}$$