3 Charge Through a Looking Glass

(idea L. Levitov, solution A. Sadofyev)

Can you engineer a dielectric permittivity tensor such that a real charge at some distance to an infinite half-space filled with a dielectric will generate an induced (real) point-like charge inside the medium? Note: a medium with such permittivity tensor might be not realizable, at least in a static (equilibrium) setup.

Solution:

Let's start with a local requirements on the electric field E and displacement D inside the medium and close to the emergent charge placed at $r = 0$:

$$\mathrm{div}E = e\delta^{(3)}(r) \quad , \quad \mathrm{div}D = 0\,. \tag{1}$$

There is no free charge by the formulation of the problem while we still want to generate Coulomb like field by bound charges. The two intensities are not independent

$$D_i = \epsilon_{ij}E_j \tag{2}$$

and the permittivity tensor is constrained to be Hermitian. Note that one may have to consider a system far from equilibrium in order to reach this freedom of an arbitrary Hermitian tensor. However, we are rather interested in an algebraic configuration and not in its realization. Combining (1) and (2) we come to the following set of equations

$$\begin{aligned} D_i = \epsilon_{ij}E_j \quad &, \quad E_i = \frac{e}{r^3}r_i \\ \mathrm{div}D = \nabla_i\epsilon_{ij}E_j = 0\,. \end{aligned} \tag{3}$$

The problem can be visualized in terms of fluxes through a spherical surfaces with centers at the position of the charge. Since the electric field is parallel to the normal of any encircling sphere (the radial direction) the problem is reduced to $\epsilon_{ij}r_ir_j = 0$.

Let's consider an antisymmetric (purely imaginary) tensor $\epsilon_{ij}(\vec{r})$ which is a solution to this constraint. There is a subtlety in this solution however. It is related to the so-called "Hairy ball theorem", which states that there can't exist a non-vanishing, non-singular vector field tangent to a sphere at each point. More explicitly, if ϵ_{ij} is antisymmetric, then from definition of D_i it follows that $D_iE_i = \epsilon_{ij}E_iE_j = 0$, so that D and E are perpendicular. But since E is directed radially that simply means that D is tangent to any given sphere enclosing the origin, at all points. Then from the Hairy Ball theorem it means that D must have a zero or a singularity somewhere on the sphere. Since this special point (zero or singularity) exists for any radius of the enclosing sphere, the special points form some sort of a linear defect.

Since no participants followed this route, however, we leave out the details of extending this solution to take boundary conditions into account. Note that one may also follow more algebraic approaches to find the explicit form of ϵ_{ij}.