1 Three-body problem

(idea by A. Phan)

Consider particles of the same mass m, with a pairwise interaction given by a repulsive potential proportional to the inverse square of the distance between them:

$$V_{ij} = \frac{A}{(r_i - r_j)^2} \quad , \quad A > 0$$

1. For a two-body problem with initial positions and velocities given by

$$r_1 = (0,\ 0,\ -d) \quad , \quad v_1 = (0,\ 0,\ v)$$
$$r_2 = (0,\ 0,\ 0) \quad , \quad v_2 = (0,\ 0,\ 0)$$
$$d > 0 \quad , \quad v > 0$$

   you are asked to find the particle trajectories.

2. Now, for a three-body problem with an additional particle

$$r_3 = (0,\ 0,\ d) \quad , \quad v_3 = (0,\ 0,\ v)$$

   find the final velocities.

Solution:

1) The trick here is that a particle of mass μ moving in an effectively 1d space with potential:

$$V(x) = \frac{A}{x^2}$$

Such description is equivalent to the same particle but moving in an effectively 2d space with no potential at all but with an angular momentum:

$$L = \sqrt{2\mu A}$$

So by adding an extra dimension, this problem can be solved without dealing with the ODE.
2) The trick here is to find the 3 integrals of motion. There are many ways to do that, but the most systematic one comes from the Lax formalism. The Hamiltonian of the (effectively 1d space) system:

$$H = \sum_{j=1}^{3} \frac{p_j^2}{2m} + \sum_{j<k} \frac{A}{(z_j - z_k)^2}$$

This system admits a Lax representation with the symmetric operator $\hat{L}$:

$$\hat{L} = \begin{pmatrix} \frac{p_1}{\sqrt{m}} & \frac{i\sqrt{A}}{z_1 - z_2} & \frac{i\sqrt{A}}{z_1 - z_3} \\ \frac{i\sqrt{A}}{z_2 - z_1} & \frac{p_2}{\sqrt{m}} & \frac{i\sqrt{A}}{z_2 - z_3} \\ \frac{i\sqrt{A}}{z_3 - z_1} & \frac{i\sqrt{A}}{z_3 - z_2} & \frac{p_3}{\sqrt{m}} \end{pmatrix}$$

Conservations can then be read-off:

$$\mathrm{Tr}\hat{L} = \sum \frac{p_j}{\sqrt{2m}}$$

$$\mathrm{Tr}\hat{L}^2 = 2\left(\sum \frac{p_j^2}{2m} + \sum_{j<k} \frac{A}{(z_j - z_k)^2}\right)$$

$$\det \hat{L} = \prod_{j}^{3} \frac{p_j}{\sqrt{2m}} - \frac{A}{\sqrt{2m}} \left(\frac{p_1}{(z_2 - z_3)^2} + \frac{p_2}{(z_3 - z_1)^2} + \frac{p_3}{(z_1 - z_2)^2} \right)$$

For final velocities calculation, we get 3 equations for 3 unknowns $v_1 < v_2 < v_3$:

$$v_1 + v_2 + v_3 = v$$
$$v_1^2 + v_2^2 + v_3^2 = v^2 + \frac{9A}{2md^2}$$
$$v_1 v_2 v_3 = -\frac{2Av}{md^2}$$