5 Fermionic lobsters

(idea by F. Popov)

Peter The Experimentalist caught a finite number of lobsters in Maine. He decided first to play with them before having a nice dinner. Peter put these lobsters on an one-dimensional lattice. It is forbidden for lobsters to be at the same site of the lattice at the same moment. After that Peter waits N minutes, during a minute only one lobster can make only one hop to the left or to the right. Peter wants to know the possible final states with their multiplicities if the lobsters are allowed to hop only to the right $\left\{n_i^R\right\}$ or to the left $\left\{n_i^L\right\}$. You are asked to prove or disprove the following statements:

1. $\sum n_i^L = \sum n_i^R$
2. $\sum \left(n_i^L\right)^2 = \sum \left(n_i^R\right)^2$
3. If we are given a pattern of moves L or R and also the possible final states with multiplicities $\left\{n_i^{\text{arb}}\right\}$, then $\sum \left(n_i^{\text{arb}}\right)^2$ doesn't depend on the choices L or R but on the number of hops.

Solution

We introduce a set of creation-annihilation operators $\{c_i\}_{i=-\infty}^{\infty}$ satisfying the relations

$$\left\{c_i, c_j^\dagger\right\} = 2\delta_{ij} \tag{1}$$

The corresponding Hilbert space are just an infinite sequence of 0 and 1, e.g $|\ldots 0011100\ldots\rangle$. The possible outcomes can be represented as a linear combinations of such states. It can be constructed out of any initial state by applying two hamiltonians

$$H_R = \sum c_{i+1}^\dagger c_i, \quad H_L = \sum c_{i-1}^\dagger c_i \tag{2}$$

One can check that

$$[H_L, H_R] = 0, \quad H_L^\dagger = H_R. \tag{3}$$

From this immediately follows that

$$\sum n_i^{R,2} = \langle in|H_L^N H_R^N|in\rangle = \langle in|H_R^N H_L^N|in\rangle = \sum n_i^{L,2} \tag{4}$$

From this one can infer that we can choose any sequence of L and R and get the same number of $\sum n_i^{S,2}$.

To prove the first statement, we take a finite size interval I where the lobsters at any final state. We consider the following state

$$|univ\rangle = \sum_{i=\ldots 000\ldots}^{\ldots 111\ldots} |i\rangle \tag{5}$$

One can see that $H_R\,|univ\rangle = H_L\,|univ\rangle$ therefore

$$\sum n_i^L = \langle univ|H_L^n|in\rangle = \langle univ|H_R^n|in\rangle = \sum n_i^R \tag{6}$$