ITPO 2019, Problem 1: Bendy Plates*

(Dated: March 5, 2019)

PROBLEM STATEMENT

A culinarily challenged amplitudologist decided to bake some cookies. To his horror, he discovered that his baking sheet has spontaneously broken a $\mathbb{Z}_2$ symmetry: a formerly at metal sheet, it would bulge up and down, picking one of the two vacua under a slight application of pressure.

Why would a metal sheet bulge like that? Why was the symmetry broken? Devise a toy model for the baking sheet that would exhibit this symmetry-breaking property. Make whatever assumptions you finnd reasonable.

GRADING RUBRIC

	Criterion	Point value	Total
Explains the physical phenomenon	Understands terms used	1	5
	Robust to change in assumptions	3	
	Overall quality	1	
Offers a viable toy model	Captures SSB behavior	3	15
	Generalizable	1	
	Derives SSB within the toy model	8	
	Shows Understanding of SSB	2	
	Overall quality	1	
			20

SOLUTION

The process of bulging is rooted in the thermal expansion (contraction) of the bulk of the baking sheet that has a relatively less pliable boundary. Assume that at room temperature, the baking sheet is flat — that is, the equilibrium perimeter of its boundary and the equilibrium area of its bulk are such that the flat configuration is the one with the lowest energy. The equilibrium is stable due to elastic forces that act to keep the sheet in its flat configuration.

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During heating or cooling, the equilibrium area of the bulk may expand (or shrink) at a different rate than the equilibrium perimeter of the boundary. If the energy cost of this mismatch is large enough, the strain associated with this differential thermal expansion exceeds the curvature-dependent strain that keeps the baking sheet flat.

To understand the process, it is sufficient to consider a one-dimensional example (a rod). Setting aside thermal expansion, suppose that the potential energy of the rod is well-described by an elasticity-like term,

$$U_{\mathrm{e}}\left[y(x)\right] = \int_0^{L_0} \alpha \left(\frac{\partial y}{\partial x}\right)^2 + \beta \left(\frac{\partial^2 y}{\partial x \partial x}\right)^2 \mathrm{d}x \,, \tag{1}$$

Here, odd powers in the expansion are absent by parity arguments ($y \to -y$), and α and β have the same sign. L_0 is the length of the rod at room temperature. This potential energy term results in elastic forces that keep the baking sheet flat at room temperature.

Now, consider the thermal expansion of the sheet. Suppose that the boundary is completely inflexible. In a two-dimensional model, this would imply a boundary with a constant perimeter; in this toy one-dimensional model, we simply keep endpoints fixed. Finally, let us add the potential energy term corresponding to the thermal expansion of the bulk. We write it in the form of a Lagrange multiplier term that fixes the length of the rod to be $L(T)$. The length is determined by thermal expansion and coincides with L_0 at room temperature:

$$U_{\mathrm{L}}\left[y(x)\right] = -\lambda(T) \left(\int_0^{L_0} \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \right) \mathrm{d}x - L(T) \right) \tag{2}$$

For small deviations $\frac{\partial y}{\partial x}$, we may write

$$U_{\mathrm{L}}\left[y(x)\right] \approx -\lambda(T) \left(\int_0^{L_0} \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x - \Delta L(T) \right) . \tag{3}$$

Collecting terms in this approximation, we write the potential energy of the rod as

$$U\left[y(x)\right] = \lambda(T)\Delta L(T) + \int_0^{L_0} (\alpha - \lambda(T)) \left(\frac{\partial y}{\partial x}\right)^2 + \beta \left(\frac{\partial^2 y}{\partial x \partial x}\right)^2 \mathrm{d}x \tag{4}$$

The term outside the integral is irrelevant and at any given temperature, can be eliminated by shifting the potential energy. To see how the remaining expression is explicitly symmetry-breaking, let us do a Fourier transform, setting aside the question of fixed boundary conditions by concentrating on the bulk far away from the edges as well as setting a reasonable UV cutoff.

Under these assumptions, we can re-write the integrand of the Fourier-transformed expression as

$$\mathcal{U}\left[\tilde{y}(k)\right] = \left((\alpha - \lambda(T))k^2 + \beta k^4\right)\left(\tilde{y}(k)\exp\left(ikx\right)\right)^2 \tag{5}$$

If the forces associated with the thermal expansion are large enough ($\lambda(T) > \alpha$), the quadratic term will change sign, resulting in a familiar expression of a Mexican hat potential. The ground state of the system is then degenerate. The two minima in momentum space translate into two real-space configurations (bulging up or down). The flat configuration becomes a point of unstable equilibrium, and any slight mismatch of pressures drives the system into either of the two minima, breaking the $\mathbb{Z}_2$ symmetry.

This official solution is based on the excellent solutions provided by teams *JadWIN*, *Fadeev Popov Ghosts*, and *Cartoon Subalgebra*.[1–3]. Well done!

[1] Team Cartoon Subalgebra, *Problem 1. Bendy Plates* (2019).
[2] Team Fadeev Popov Ghosts, *Problem 1: Bendy Plates* (2019).
[3] Team JadWIN, *Solution to problem 1* (2019).