ITPO 2020 Solutions

Problem 2

Exercise (*2*): Wiley Invariance.

Consider a free, massless scalar field ϕ in two-dimensional Minkowski spacetime. Show that the action is invariant under the transformation rule

$$\phi \mapsto \phi + \delta\phi \equiv \phi - \varepsilon(z)\partial\phi - \bar{\varepsilon}(\bar{z})\bar{\partial}\phi \tag{1}$$

Where z and $\bar{z}$ are the so-called "lightcone coordinates" in two dimensions, ∂ and $\bar{\partial}$ are their corresponding derivatives, and $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are small but arbitrary functions of the indicated coordinates. Next, for a small parameter α consider the following Lagrangian density, as an extension of the free scalar:

$$\mathcal{L} = \partial\phi\bar{\partial}\phi + \alpha(\partial\phi\bar{\partial}\phi)^2 \tag{2}$$

Is it possible to add $\mathcal{O}(\alpha)$ contributions to $\delta\phi$ which are first-order in ϕ-derivatives and which preserve this invariance?

Rubric:

Criterion	Point value	Total
Demonstrates initial invariance	2	20
States correctly that contributions are not possible	3	
Correct proof	13	
Correct analysis of significance	2	

Solution:

The first part of the problem is a textbook exercise. Let us replace $\phi \mapsto \phi + \delta\phi$ in $\mathcal{L}$ and consider only the terms which are first order in ε. Without loss of generality we will discard $\bar{\varepsilon}$ terms, because the analysis will be identical (due to the Z_2 symmetry of the action under $\partial \leftrightarrow \bar{\partial}, \varepsilon \leftrightarrow \bar{\varepsilon}$). We find:

$$\begin{aligned}\delta\mathcal{L} &= -\partial(\varepsilon\partial\phi)\bar{\partial}\phi - \partial\phi\bar{\partial}(\varepsilon\partial\phi) \\ &= -\partial\varepsilon\partial\phi\bar{\partial}\phi - \varepsilon\partial\partial\phi\bar{\partial}\phi - \varepsilon\partial\phi\bar{\partial}\partial\phi\end{aligned} \tag{3}$$

Now, since we are in flat space, we can discard boundary terms – integrating the first term by parts gives:

$$\delta\mathcal{L} = \varepsilon\partial\partial\phi\bar{\partial}\phi + \varepsilon\partial\phi\partial\bar{\partial}\phi - \varepsilon\partial\partial\phi\bar{\partial}\phi - \varepsilon\partial\phi\bar{\partial}\partial\phi = 0 \tag{4}$$

Which establishes invariance of the original action under these transformations, which are called "Weyl transformations."

For the second part of the problem, decompose $\mathcal{L}$ and $\delta\phi$ order-by-order in α:

$$\mathcal{L} = L^{(0)} + \alpha L^{(1)} + \mathcal{O}(\alpha^2) \tag{5}$$

$$\delta\phi = \delta^{(0)}\phi + \alpha\delta^{(1)}\phi + \mathcal{O}(\alpha^2) \tag{6}$$

Applying $\delta^{(0)}$ and integrating by parts, we find:

$$\int d^2z\, \delta^{(0)} L^{(1)} = \int d^2z\, \frac{1}{2}(\partial\phi\bar{\partial}\phi)^2 \left[\partial\varepsilon + \bar{\partial}\bar{\varepsilon}\right]. \tag{7}$$

At the next step in perturbation theory, we consider that

$$\int d^2z\, \delta^{(1)} L^{(0)} = \int d^2z\, 2\partial\bar{\partial}\phi \cdot \delta^{(1)}\phi \tag{8}$$

In order for $\delta^{(1)}\phi$ to be well-defined for all ε and $\bar{\varepsilon}$, we need to equate (7) with (8), which in particular requires us to integrate (7) by parts to find a term proportional to $\partial\bar{\partial}\phi$.

There are a number of ways to show that you cannot find such a term, and therefore one cannot define $\delta^{(1)}\phi$. Intuitively, it this happens because integrating by parts any given derivative in (7) will yield terms involving both $\partial\bar{\partial}$ and either of $\partial\partial$ or $\bar{\partial}\bar{\partial}$. Exhausting all the possibilities, one formally proves that such additions to the transformation rules cannot be formulated.

This order-α addition to the Lagrangian density is for various reasons called the $T\bar{T}$ deformation of the massless, free scalar field. Such deformations are known not to preserve Weyl invariance, which is what we have basically shown here.