ITPO 2020 Solutions

Problem 3

Exercise (*3*): Swiss Cheese Universe.

Write down a spacetime metric corresponding to a classical black hole system in an infinite number of spacial dimensions. Using your result, generate what is known as a *multi-center* black hole geometry. More specifically, describe an infinite-dimensional spacetime which has an arbitrary number of black holes interspersed at arbitrary locations.

Now relax the limit of infinite dimensions and consider this spacetime in a large enough number of dimensions d so that the geometry is nearly identical. Distribute a number density ρ of these black holes approximately uniformly across some d-sphere of finite radius R. Analytically or numerically, model the density of a diffusive gas of massive particles as it moves through this spacetime, provided the initial density of the gas is Gaussian-distributed with width $\ell \ll R$ and a small total mass M.

Rubric:

	Criterion	Point value	Total
Part A:	Wrote down a sensible metric in d=inf dimensions	1	10
	Correct argument for multi-center geometry	4	
	Wrote down appropriate diffusion equation	5	
Part B:	Correct numerical or analytic results	9	10
	Correct analysis of solution significance	1	

Solution:

Following [2], the Schwarzschild metric in $D \geq 3$ space-time dimensions is:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}(dr^2 + r^2 d\Omega_{D-2}^2), \qquad f(r) = 1 - \left(\frac{r_s}{r}\right)^{(D-3)} \tag{1}$$

Where $d\Omega_{D-2}$ defines the surface metric of the $(D-2)$-sphere and r_s defines the Schwarzschild radius of the black hole. In terms of the mass M sourcing the geometry, r_s takes the following form:

$$r_s^{(D-3)} = \frac{2GM}{c^2} \cdot \frac{2\Gamma\left(\frac{D-2}{2}\right)}{(D-2)\pi^{(D-4)/2}} \tag{2}$$

Note that as $D \to \infty$, this radius technically diverges. However, we can keep it finite by tuning M in the limit. As long as we keep in mind that the mass must be parametrically small in order to prevent this divergence in the limit of large D, we are safe for the purposes of this problem.

Assume r_s is nonzero and constant and set $c = 1$. If we take the $D = \infty$ limit of the above geometry, we see three crucial regions arise:

$$f(r) = \begin{cases} 1 & r > r_s, \qquad \text{Flat space} \\ 0 & r = r_s, \qquad \text{Horizon} \\ -1 & r < r_s, \qquad \text{Flat interior} \end{cases} \tag{3}$$

The black hole essentially carves out of flat space an ∞-sphere of radius r_s. This solution suggests that we are free to place such defects wherever we please, provided that they do not overlap. The corresponding metric looks like:

$$f(\vec{x}) = \prod_{i=1}^{N} \Big[\Big(1 - 2\Theta\big(r_{s,i} - |\vec{x} - \vec{x}_i| \big) \Big) \Big(1 - \delta_{|\vec{x}-\vec{x}_i|, r_{s,i}} \Big) \Big] \tag{4}$$

Where $\vec{x}$ denotes the spatial location, $\vec{x}_i$ the center of each of N black holes, and $r_{s,i}$ the radius corresponding to the i^{th} black hole.

Now, we relax the infinite-D limit and introduce the proposed mass density. There are many valid approaches to this problem as stated, e.g. using Fick's law. Correct approaches with such methodologies were able to receive full points for this problem.

However, there is an interesting alternative, which we elaborate upon below. The assumptions of the problem make it reasonable to neglect the effects of backreaction, so we are effectively studying the diffusion of matter in a porous medium. This behavior is well-studied in the physics of aquifers, and an interesting solution space suggests itself if we expect the matter density to obey the *modified porous medium equation* (MPME):

$$\partial_t \rho = \kappa \nabla^2 \rho^{1+n} \tag{5}$$

Where n is a parameter we should fix through physical considerations, and the transport coefficient κ has a nontrivial dependence on the time derivative of the density:

$$\kappa = \frac{\ell^{(D-1)n+1}}{M^n} \cdot \begin{Bmatrix} 1, & \partial_t \rho > 0 \\ 1 + \epsilon, & \partial_t \rho < 0 \end{Bmatrix} \tag{6}$$

This relation differs from the standard *porous medium equation* by this nontrivial behavior of κ. The intuition of $\epsilon \neq 0$ is that the pores of the aquifer release fluid at a different rate than they absorb it.

The solutions to the MPME for radially symmetric Gaussian initial conditions were studied in [1]. For large D and long times, we expect the solution to take the form

$$\rho(\vec{r}, t) \sim \frac{M}{\ell^{(D-1-1/n)}} \frac{1}{t^{1/n}} f\left(\frac{r \ell^{1/(nD)-1}}{t^{1/(nD)}}, \epsilon \right) \tag{7}$$

Solving for features of the dimensionless f, numerically or analytically, is considered an open area of research.

References

[1] Lin-Yuan Chen, Nigel Goldenfeld, and Y. Oono. Renormalization-group theory for the modified porous-medium equation. *Phys. Rev. A*, 44:6544–6550, Nov 1991.

[2] Chang Jun Gao. Arbitrary dimensional Schwarzschild-FRW black holes. *Class. Quant. Grav.*, 21:4805–4810, 2004.