ITPO 2021 Solutions

Problem 3

Exercise (*3*): Cut to the trace.

Consider the following matrix path integral:

$$\mathcal{Z} = \int \prod_{i,j} dM_{ij}\, e^{-\frac{N}{2}\mathrm{Tr}\,(M^2)+N^2 V(M)} \tag{1}$$

where the integration is over the elements of an $N \times N$ Hermitian matrix and $V(M)$ is an arbitrary potential.

1. Take $V(M) = \frac{h}{N^2}\left(\mathrm{Tr}\, M\right)^2$, where h is a coupling constant. Calculate the probability distribution of a single eigenvalue of M, assuming N is very large.

2. Compute the same distribution assuming $V(M) = \frac{g}{N}\mathrm{Tr}\, M^{2k} + h\left(\exp\left(\frac{1}{N}\mathrm{Tr}\, M^{2k}\right) - 1\right)$. (You will receive partial credit if you solve for specific values of k).

Rubric:

	Criterion	Point value	
Part 1	Wigner Semicircle	1	
	Correct radius of semicircle	4	
Part 2	Correct algebra, but truncates exponential	2	
	Correct algebra, but fixed values of k	5	
	Correct algebra	15	

Solution:

Part (1) of this exercise exists in the literature, so we will solve it explicitly here. Part (2) is an open problem. For a good review of random matrix theory, see [1].

Discussions of random matrix theory usually focus on single-trace potentials, which take the form $\mathrm{Tr}\, V(M)$, for some polynomial V and some $N \times N$ random diagonalizable matrix M_{ij}, with N large. These terms are convenient to consider because they have a nice description in terms of the eigenvalues λ_k of M_{ij}:

$$\mathrm{Tr}(M^k) = \sum_{i=1}^{N} \lambda_i^k \qquad \Longrightarrow \qquad V(\vec{\lambda}) \equiv \mathrm{Tr}\, V(M) = \sum_{i=1}^{N} V(\lambda_i). \tag{2}$$

The eigenvalue potential factorizes, with the eigenvalues λ_j becoming identically distributed random variables parametrized by the same potential $V(\lambda_i)$.

In general, however, it is not necessary to restrict $V(M)$ to single-trace interactions. In this problem, we considered a potential which contains terms with $\mathrm{Tr}(M^k)^n$ interactions, which are of order $k \cdot n$ in eigenvalues. The multi-trace case has been considered in the literature before, but sparsely. It is tragically easy to explain why, since:

$$\mathrm{Tr}(M^k)^n = \sum_{\sum_i p_i = n} \left(\prod_{j=1}^{N} \frac{j \lambda_j^{k \cdot p_j}}{p_j!} \right). \tag{3}$$

The eigenvalues are still identically distributed,[1] but the potential no longer decomposes into a sum of single-eigenvalue potentials. If our starting point is the partition function of the Gaussian Unitary Ensemble (GUE), the simplest effective action we can write is

$$Z_{GUE} = \int \prod_{i=1}^{N} d\lambda_i e^{-N^2 S(\vec{\lambda})}, \qquad S(\vec{\lambda}) = \frac{1}{N} V(\vec{\lambda}) - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j|. \tag{4}$$

We can borrow much of our notation from the single-trace case. The key difference is that our potential is now a symmetric polynomial of the λ_i which cannot be expressed as $\sum_i V(\lambda_i)$. We should check that the usual definitions still behave in the way we expect. The saddle point approximation gives us the following condition on the $\tilde{\lambda}_i$ at a given saddle:

$$\frac{2}{N} \sum_{j \neq i} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j} = \partial_{\lambda_i} V(\vec{\lambda})\big|_{\vec{\lambda} = \vec{\tilde{\lambda}}} \tag{5}$$

For sanity's sake, we will drop the tildes on the λ_i, as it will be clear when we are at a saddle. The resolvent $G(x)$ must also be adapted slightly, in particular because we would like to relate it to the probability density of a single eigenvalue. We define the densities ρ_i obtained after marginalizing out $N - i$ eigenvalues:

$$\rho_i(\mu_1, \ldots, \mu_i) = \int \prod_{j=N-i+1}^{N} d\mu_j \, \rho_N(\mu_1, \ldots, \mu_N) \tag{6}$$

This definition ensures that

$$G(x) = \int d^N \mu \, \rho_N(\vec{\mu}) \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{x - \mu_j} \right) = \int d\mu \, \frac{\rho_1(\mu)}{x - \mu}, \tag{7}$$

and thus the discontinuity of the resolvent around the branch cut on its support is defined in terms of the marginalized density

$$G(x + i\varepsilon) - G(x - i\varepsilon) = -2\pi i \rho_1(x). \tag{8}$$

[1] This is obvious starting from (2), but it is also immediate in (3) from the sum over the partitions of n, which are the characteristic classes of the symmetric group of n elements.

It will also be useful to define different components of the potential,

$$V_{k,n}(\vec{\lambda}) = \frac{1}{n!k!}\frac{1}{N^{n-1}}\left(\sum_{i=1}^{N}\lambda_i^k\right)^n \qquad \Longrightarrow \qquad \partial_{\lambda_i}V_{k,n}(\vec{\lambda}) = \frac{1}{N}\lambda_i^{k-1}V_{k,n-1}(\vec{\lambda}). \tag{9}$$

The scaling with N is chosen so that when the λ_i are of order 1, the argument to the exponent in the path integral is of order N^2. In other words, multi-trace should be thought of as multi-$\frac{1}{N}$Tr so that those contributions compete with single-trace terms in the potential at large N.

It is worth understanding why a standard treatment of the problem does not immediately work when we include a multi-trace potential. Note the following:

$$\int d^{N-1}\lambda\,\rho_{N-1}(x,\vec{\lambda})\left(\frac{2}{N}\sum_{j=2}^{N}\frac{1}{x-\lambda_j}\right) = \int d^{N-1}\lambda\,\rho_{N-1}(x,\vec{\lambda})\partial_x V(x,\vec{\lambda}) \tag{10}$$

Since ρ_{N-1} is a symmetric function of its arguments, we can rearrange this expression in the large-N limit to find that for x on the branch cut:

$$2\,\text{P.P.}\int d\lambda\,\frac{\rho_1(x,\vec{\lambda})}{x-\lambda} = \int d^{N-1}\lambda\,\rho_{N-1}(x,\vec{\lambda})\partial_x V(x,\vec{\lambda}) \tag{11}$$

In the single-trace limit the RHS is just $V'(x)$, which is (2.16) in Zuber [1]. The fact that this expression no longer simplifies directly is at the heart of the difficulty with multi-trace potentials. To understand why, re-derive (2.12) in Zuber. We see a difference in the fourth line there:

$$G(x)^2 = -\frac{1}{N}G'(x) + \frac{1}{N}\sum_{i=1}^{N}\frac{\partial_{\lambda_i}V(\vec{\lambda})}{x-\lambda_i} \tag{12}$$

There is no longer a way to simplify the RHS by adding and subtracting a term which pulls out of the sum (i.e. $V(x)$). To make progress, we need to find a self-consistent way of introducing the moments of ρ_1 into the analysis. We will show how to achieve this for our multi-trace potential.

In line with the results of original interest in the literature [2, 3, 4], we will primarily consider a potential of the form

$$V(\vec{\lambda}) = V_{2,1}(\vec{\lambda}) + 6gV_{4,1}(\vec{\lambda}) + hV_{2,2}(\vec{\lambda}). \tag{13}$$

With $h = 0$ this reduces to a sum over i of $V(\lambda_i) = \frac{1}{2}\lambda_i^2 + \frac{g}{4}\lambda_i^4$. We find that

$$\partial_{\lambda_i}V(\vec{\lambda}) = \lambda_i\Bigg(1 + h\cdot\underbrace{\frac{1}{N}\sum_{j=1}^{N}\lambda_j^2}_{c_2}\Bigg) + g\lambda_i^3 \equiv A(\lambda_i). \tag{14}$$

Note that, in moving to the second equality, we have assumed that c_2 is constant, so that the sum over λ^2 only shifts the coefficient of λ^1 in the saddle point equation. The authors

of [2] justify this step by large-N factorization, and it motivates an ansatz whereby we can solve for the large-N statistics by replacing any sums over λ_i with their expectation values, c_i. Here, the large-N description will be valid and self-consistent so long as we impose that c_2 be the second moment of the ρ_1 that we derive. This procedure will in principle work for a generic multi-trace potential, because every time we fix one of the moments c_i, we also impose a single constraint, i.e. that $\int d\lambda\, \lambda^i \rho_1(\lambda) = c_i$. The computational details will differ case-by-case, but a consistent solution will exist so long as ρ_1 can be assumed to have finite support.

Now, we can apply the usual resolvent analysis and find

$$G(x) = \frac{1}{2}\left(A(x) - \sqrt{A(x)^2 - 4P(x)}\right) \equiv \frac{1}{2}A(x) - Q(x)\sqrt{x^2 - 4a^2}, \tag{15}$$

where

$$P(x) = \frac{1}{N}\sum_{i=1}^{N} \frac{A(x) - A(\lambda_i)}{x - \lambda_i}. \tag{16}$$

Expanding $Q(x) = \sum_{i=0}^{\infty} q_i x^i$ and studying large $|x|$, we find

$$G(x) = \frac{1}{2}(g - 2q_2)x^3 - q_1 x^2 + \left(\frac{1}{2} + \frac{1}{2}c_2 h - q_0 + 2a^2 q_2\right)x + 2a^2 q_1 + \frac{2a^2}{x}(q_0 + a^2 q_2) + \mathcal{O}(x^{-2}). \tag{17}$$

Imposing $G(x) \sim 1/|x|$ gives constraints on $Q(x)$, i.e.

$$q_0 = \frac{1}{2} + \frac{1}{2}c_2 h + g a^2, \qquad q_1 = 0, \qquad q_2 = \frac{g}{2}. \tag{18}$$

So, the resulting density looks like

$$\frac{1}{2\pi}\left(1 + c_2 h + 2g a^2 + g x^2\right)\sqrt{4a^2 - x^2}. \tag{19}$$

We just have to get rid of the factors of c_2. To do so, we impose self-consistency conditions on the distribution. Namely,

$$1 = \int_{-2a}^{2a} d\lambda\, \rho_1(\lambda), \qquad\qquad c_2 = \int_{-2a}^{2a} d\lambda\, \lambda^2 \rho_1(\lambda). \tag{20}$$

Insodoing, we can finally write down the end result:

$$\rho_1(\lambda) = \frac{1}{2\pi}\left(\frac{1}{a^2} - g a^2 + g\lambda^2\right)\sqrt{4a^2 - \lambda^2}, \tag{21}$$

where a must satisfy

$$g h a^8 + (h + 3g)a^4 + a^2 - 1 = 0. \tag{22}$$

Notably, when $h = 0$ we recover Zuber's (2.20)-(2.21). When $g = 0$, we find Wigner semicircle behavior, but with $a = a(h) \neq 2$. We confirm this behavior numerically in Figure 2. Turning on both couplings, the results are less straightforward to interpret.

Numerics

We can confirm these results with a numerical analysis. To simulate the distribution of eigenvalues, we employ the Metropolis-Hastings algorithm. The i^{th} step out of n_{MH} total steps is sampled from a uniform distribution on $[-\delta, \delta]^N$ and rescaled by an exponential dampening factor $e^{-Ki/n_{MH}}$. The dampening ensures that we accept a large number of large Metropolis steps in the beginning, and a large number of small Metropolis steps toward the end of the optimization. In the results below, we have chosen to study $N = 100$, $\delta = 0.25$, $K = 10$ and $n_{MH} = 1 \times 10^5$. The number of samples is 100 for all figures. The step acceptance rate has been tuned to $\sim 50\%$, which is how we determined K and δ. Here is the Matlab script:

```
N    = 100;          % Define dimension of matrix
num = 100;          % Define number of matrices to generate
nsteps = 200000;    % Define number of Metropolis steps

% Pull seed eigenvalues from some distribution
eigs  = GenerateInitialEigs(N,num);

for imat = 1:num
    % Compute the action for this sample
    teig = eigs(:,imat);
    tprb = -V(N,diag(teig))+sum(log(abs(teig-teig')+eye(N)),"all");

    % Generate the Metropolis steps
    steps = (rand(N,nsteps)-0.5)*0.5;

    % Start Metropolis-Hastings
    for tstp = 1:nsteps
        % Generate the step and re-compute the action.
        % (Note that the steps start large and decay exponentially).
        tteig = teig + steps(:,tstp)*exp(-10*tstp/nsteps);
        ttprb = -V(N,diag(tteig))+sum(log(abs(tteig-tteig')+eye(N)),"
all");

        % Accept if step is more probable or wins weighted coin toss
        if ttprb > tprb || rand() < exp(ttprb-tprb)
            eigs(:,imat) = tteig;
            teig=tteig;
            tprb=ttprb;
        end
    end
 end

 % Plot a histogram of the results
 h = histogram(reshape(eigs,[],1),100);
```

To test that this algorithm works, we checked against the analytic large-N results for $V(M) = \frac{1}{2}M^2 + \frac{g}{4}M^4$. The checks are presented in Figure 1, and they confirm the validity of our procedure. Figure 2 demonstrates the behavior of interest.

We can qualitatively understand the double-trace interactions from these plots by appealing to a cost-benefit analysis. When N is large, the extra trace associates a cost to the size

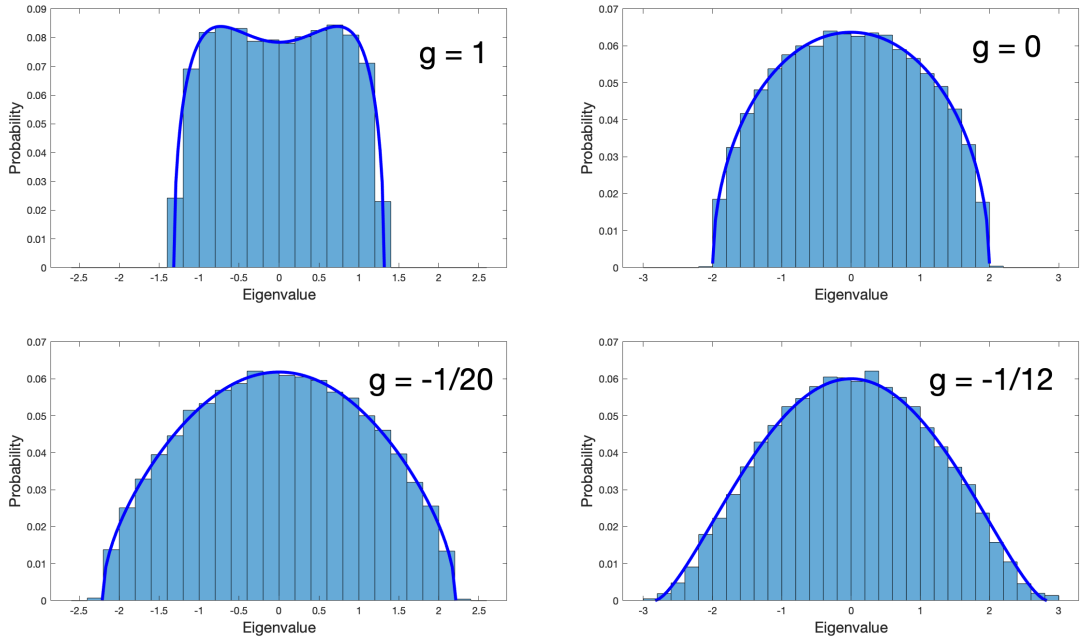

Figure 1: Checks of the numerical algorithm with $V(M) = \frac{1}{2}M^2 + \frac{g}{4}M^4$. The histogram comes from our numerics, and the curve is the analytic result in Zuber [1]. The "probability" on the y-axis is the binned probability, related to the pdf normalization by a factor of the bin width.

of a particular moment of the distribution – in this case, the second moment, which is equivalent to the variance. In our conventions, when g is large and positive, a large variance is suppressed, and the eigenvalues are pressed toward their mean. When g is large and negative, a large variance is physically favorable. This is directly visible from the scaling of a as a function of h when $g = 0$; equivalently, see Figure 2, where the Wigner semicircle simply grows or shrinks in radius depending on the value of h.

References

[1] J. Zuber, "Introduction to random matrices," *Lectures at Les Houches* (2012) .

[2] S. R. Das, A. Dhar, A. M. Sengupta, and S. R. Wadia, "New Critical Behavior in $d = 0$ Large N Matrix Models," *Mod. Phys. Lett. A* **5** (1990) 1041–1056.

[3] G. Korchemsky, "Matrix model perturbed by higher order curvature terms," *Mod. Phys. Lett. A* **7** (1992) 3081–3100, arXiv:hep-th/9205014.

[4] L. Alvarez-Gaume, J. Barbon, and C. Crnkovic, "A Proposal for strings at D > 1," *Nucl. Phys. B* **394** (1993) 383–422, arXiv:hep-th/9208026.

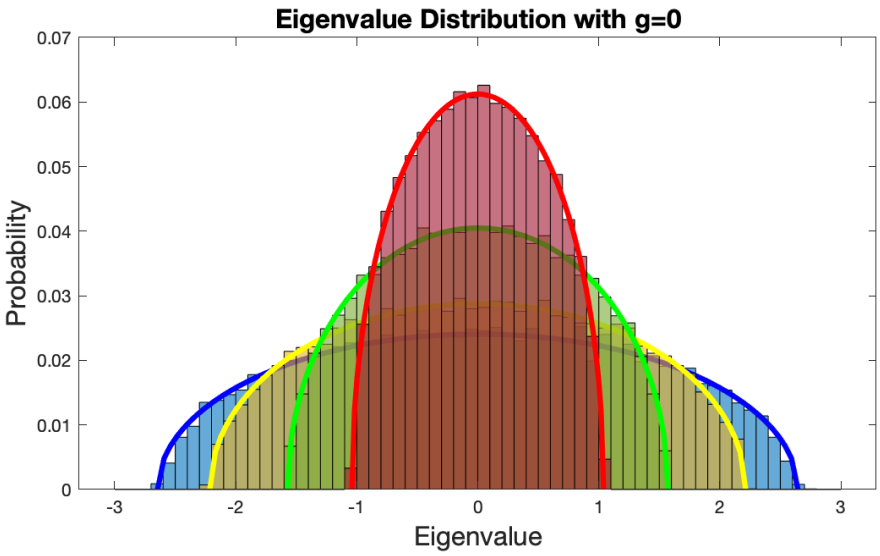

Figure 2: A plot of $\rho_1(\lambda)$ for $g = 0$, varying h. The distribution is simply the Wigner semicircle, but with $a = a(h)$. From narrowest to widest, the values of h are $10, 1, -0.15, -0.25$.