ITPO 2021 Solutions

Problem 5

Exercise (*5*): The Pursuit of Maximal Happyness.

The animal world of the Flatland (a reference to the novel by E.A. Abbott about a 2D country) is quite versatile. Take, for example, the myriapods. These creatures consist of a body with N segments, $N \gg 1$. Each segment is a piece of a straight line of length δ. In the middle of each segment there is one leg, a straight interval of length $\frac{a}{2}$, such that $N\delta \gg \frac{a}{2} \gg \delta$. Legs are always orthogonal to the segments and can go from one side of the segment to another if needed. In Flatland, myriapods can pass through each other like ghosts.

Myriapods prefer to sit straight. Their happiness level depends on the angle $\theta_{i,i+1}$ between the consecutive segments as $\sum_i \kappa \cos\theta_{i,i+1}/\delta$. They are also very extroverted animals. Namely, they like to hold each others' legs so that the ends of their legs coincide, forming a straight piece of length a. Their happiness level has a contribution proportional to the number of held legs with constant $\mu\delta$. Assume $N\delta \gg \sqrt{\frac{\kappa}{\mu}} \gg \delta$.

Consider a system of two myriapods. Obviously, the state of maximal happiness is when both myriapods are perfectly straight and hold all legs. However, due to mistakes upon their meeting they may end up in a metastable state, with the happiness level lower than a maximal possible, but such that the transition from this state to the state of maximal happiness would require an initial decrease in happiness by an amount proportional to the total number of segments.

1) Classify the possible metastable states (defects). When analyzing defects which are observable even when $a = 0$, you may simplify and set $a = 0$. Otherwise, keep a nonzero.

2) Consider pure defects of each type, such that effects of other defect types are not present. How many independent parameters does each defect have? Do these parameters completely determine the shape of the myriapods?

3) In each class of defect, determine the angle between directions of the myriapods at infinity, both on the left and on the right of the defect, as a function of the defect parameters, as well as κ and μ.

4) One class of defects demonstrates a phase transition when parameters of the problem

are varied while the parameters of the defects are fixed. Identify this class, as well as the behavior in the vicinity of the phase transition. You may solve this part of the problem numerically.

Solution:

We can divide the defects into two classes based on the scaling of their size with N. The first class includes localized defects (i.e., defects as they are most commonly understood). The defect of this class involves only finite size segments of the myriapods (not proportional to N), and the happiness associated to such a defect is less than the ground state happiness by a finite amount. In such a state, two myriapods are parallel to each other before and after the defect, so if one would cut out the defect and glue myriapods in its place, it would be a ground state for shorter myriapods. There are two options:

1. In the defect myriapods do not cross, but instead a piece of one of them is longer than the corresponding piece of the other. We refer to this defect as a "loop." This defect is different from the zero-defect state in $a = 0$ limit. The free parameter of the defect is the amount of length mismatch.

2. In the defect myriapods do cross, so the one that was on top is now at the bottom. This may involve a length mismatch as well, but does not have to. We refer to this defect as a "braid." To distinguish it from a loop, we study braids with the mismatch length that maximizes total happiness. This defect coincides with the zero-defect state in $a = 0$ limit. As a pure defect, braid defect has no free parameters (only the parameters of the myriapods themselves).

The defect of the second type involves a global misalignment of myriapods. The happiness of myriapods in such a state is less than that in the ground state by an amount proportional to N. Due to their global nature, the defects of this class are significantly more varied than those of the first class; however, they all require as a a building block a trapped global mismatch between two myriapods, that we consider a pure defect. We refer to this object as an "overlap." This defect is different from the zero-defect state in the $a = 0$ limit. The free parameter of this defect is the size of the overlapped region. When it is pure, this defect does not involve any bending of myriapods, so the angles between their legs are always zero.

For analyzing the angles produced by first two types of defects, see [1], where myriapods are called "filaments," the legs are called "cross links," and happiness is energy with opposite sign. The continuous limit is taken, which is justified since δ is small, so any sharp angle between two segments significantly decreases happiness, and it will always be relaxed via a reconfiguration of a finite number of neighboring legs and segments).

References

[1] Valentin M Slepukhin, Maximilian J Grill, Qingda Hu, Elliot L Botvinick, Wolfgang A Wall, and Alex J Levine. Topological defects produce kinks in biopolymer filament bundles. *Proceedings of the National Academy of Sciences*, 118(15), 2021.